Torsion in the Khovanov homology of 3-braids

Alex Chandler
Joint with A. Lowrance, R. Sazdanović, V. Summers

North Carolina State University

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Khovanov Homology

Given a link L, and an abelian group G, the Khovanov homology of L with coefficients in G is a bigraded abelian group $H^{*,*}(L, G)$ whose graded Euler characteristic is the (unnormalized) Jones polynomial:

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \operatorname{rank} H^{i,j}(L) = \tilde{J}(L).$$

- $H^{*,*}(L)$ denotes $H^{*,*}(L, \mathbb{Z})$
- Each $H^{i,j}(L, G)$ is a link invariant
- $H^{*,*}(L)$ is a stronger link invariant than $\tilde{J}(L)$

Torsion in integral Khovanov homology

- $\mathbb{Z}_2$-torsion is abundant in Khovanov homology (experimentally)
- $\mathbb{Z}_n$-torsion for $n \neq 2$ is rare (experimentally)
- Shumakovitch showed in 2018 that homologically thin links have only $\mathbb{Z}_2$-torsion (in particular alternating links)
- Sazdanović and Przytycki (2012) conjecture that 3-braids have only $\mathbb{Z}_2$-torsion (still open)
- we will provide a partial answer to this conjecture

Torsion in homologically thin links

We say that L is *homologically thin* over G if $H^{*,*}(L; G)$ is supported on two adjacent diagonals.

e.g.

$H^{*,*}(T_{2,5}, \mathbb{Z}_2) =$

					$\mathbb{Z}_2$
				$\mathbb{Z}_2$	$\mathbb{Z}_2$
			$\mathbb{Z}_2$	$\mathbb{Z}_2$	
		$\mathbb{Z}_2$	$\mathbb{Z}_2$		
		$\mathbb{Z}_2$			
$\mathbb{Z}_2$					
$\mathbb{Z}_2$					

$H^{*,*}(T_{2,5}, \mathbb{Z}) =$

					$\mathbb{Z}$
					$\mathbb{Z}_2$
			$\mathbb{Z}$	$\mathbb{Z}$	
			$\mathbb{Z}_2$		
		$\mathbb{Z}$			
$\mathbb{Z}$					
$\mathbb{Z}$					

Torsion in homologically thin links

Theorem (Shumakovitch, 2018)

Let L be homologically thin over $\mathbb{Z}_2$. Then $H^{*,*}(L)$ has no torsion of order 2^r for $r > 1$.

The main theorem in this talk is a local version of this result.

Locally homologically thin links

We say that a link L is *locally thin in* $[i_1, i_2]$ over G if $H^{*,*}(L, G)$ is supported on two diagonals for homological gradings $i_1 \leq i \leq i_2$

e.g.

	-9			-6			-3			0
-7										$\mathbb{Z}_2$
										$\mathbb{Z}_2$
								$\mathbb{Z}_2$		
						$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
					$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$			
				$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$					
			$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$					
		$\mathbb{Z}_2$	$\mathbb{Z}_2$							
	$\mathbb{Z}_2$	$\mathbb{Z}_2$								
-25	$\mathbb{Z}_2$									

$$H^{*,*}(\Delta^8 \sigma_1^{-3} \sigma_2^3, \mathbb{Z}_2)$$
$$\Delta = \sigma_1 \sigma_2 \sigma_1 \in B_3$$

Locally thin in $[-9, -6]$ over $\mathbb{Z}_2$ (also in $[-3, 0]$)

The Main Theorem

Theorem (C., Lowrance, Summers, Sazdanović (2018))

Suppose that a link L satisfies:

1. *L is locally thin in* $[i_1, i_2]$ *over* $\mathbb{Z}_p$ *for all prime p,*
2. $\dim_{\mathbb{Q}} H^{i_1,*}(L;\mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i_1,*}(L;\mathbb{Z}_p)$ *for each odd prime p, and*
3. $H^{i_1,*}(L)$ *is torsion-free.*

Then all torsion in $H^{i,*}(L)$ *is of the form* $\mathbb{Z}_2$ *for* $i \in [i_1, i_2]$.

Proof: Similar to Shumakovitch's proof, using a relationship between the Turner, $\mathbb{Z}_2$-Bockstein, and vertical differentials.

Application: Khovanov homology of 3-strand torus links

- In 2006 Turner computed $H^{**}(T_{3,q}, \mathbb{Q})$ for all q (also done independently by Stošić in 2007)
- Actually this computation works just as well over $\mathbb{Z}_p$ for p an odd prime, so for all i, j we have

$$\dim_{\mathbb{Q}} H^{i,j}(T_{3,q}, \mathbb{Q}) = \dim_{\mathbb{Z}_p} H^{i,j}(T_{3,q}, \mathbb{Z}_p).$$

- In 2017 Benheddi computed $H^{**}(T_{3,q}, \mathbb{Z}_2)$.

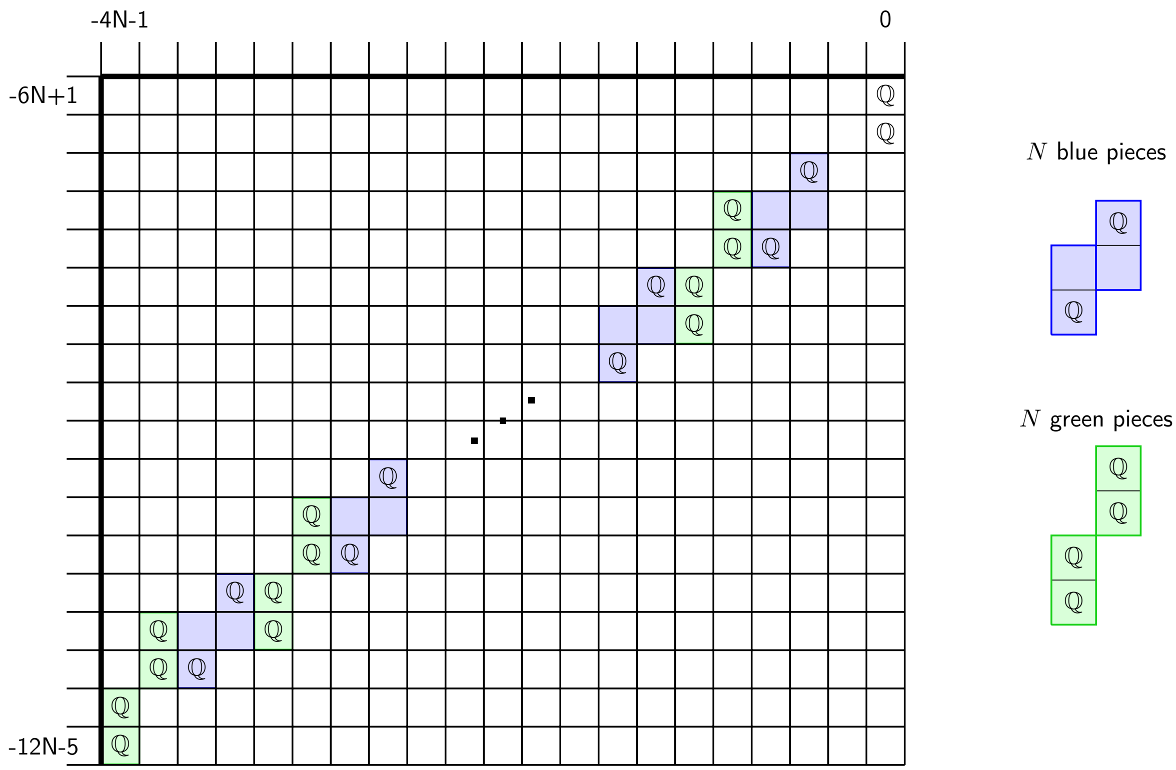


Figure: Kh($T_{3,3N+1}, \mathbb{Q}$) as computed by Turner, 2006 (Stošić, 2007)

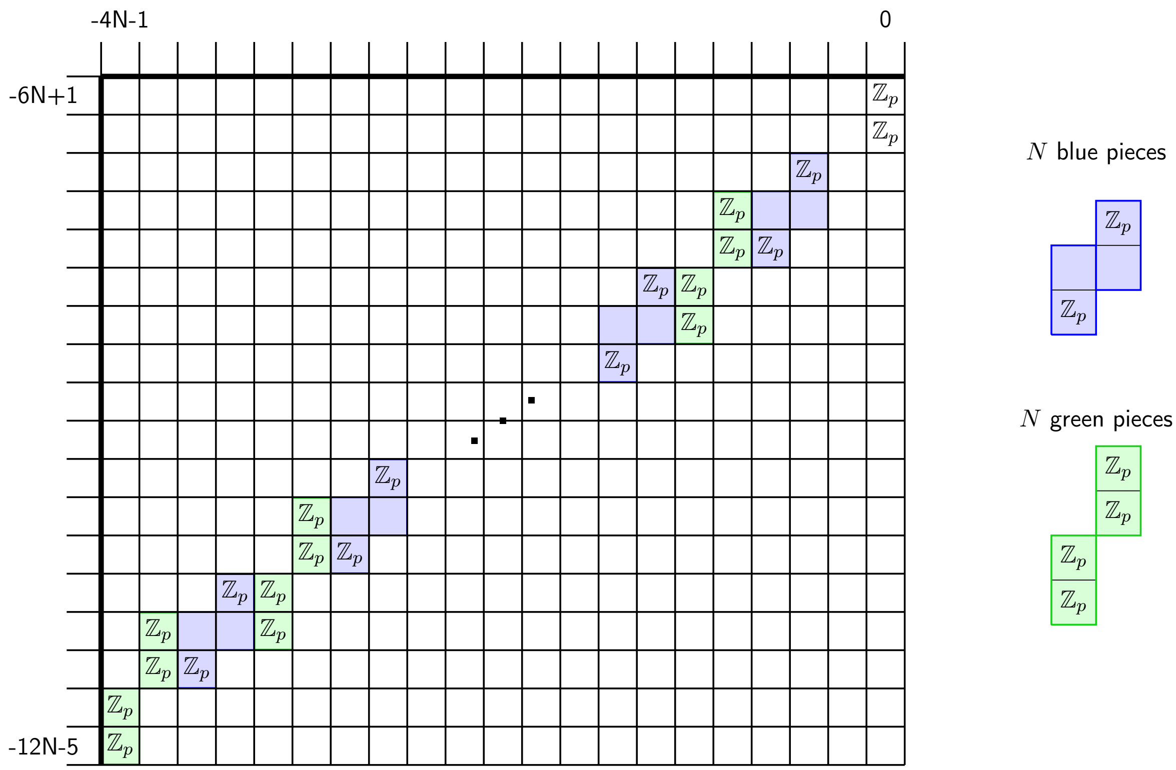


Figure: Kh($T_{3,3N+1}, \mathbb{Q}$) as computed by Turner, 2006 (Stošić, 2007)

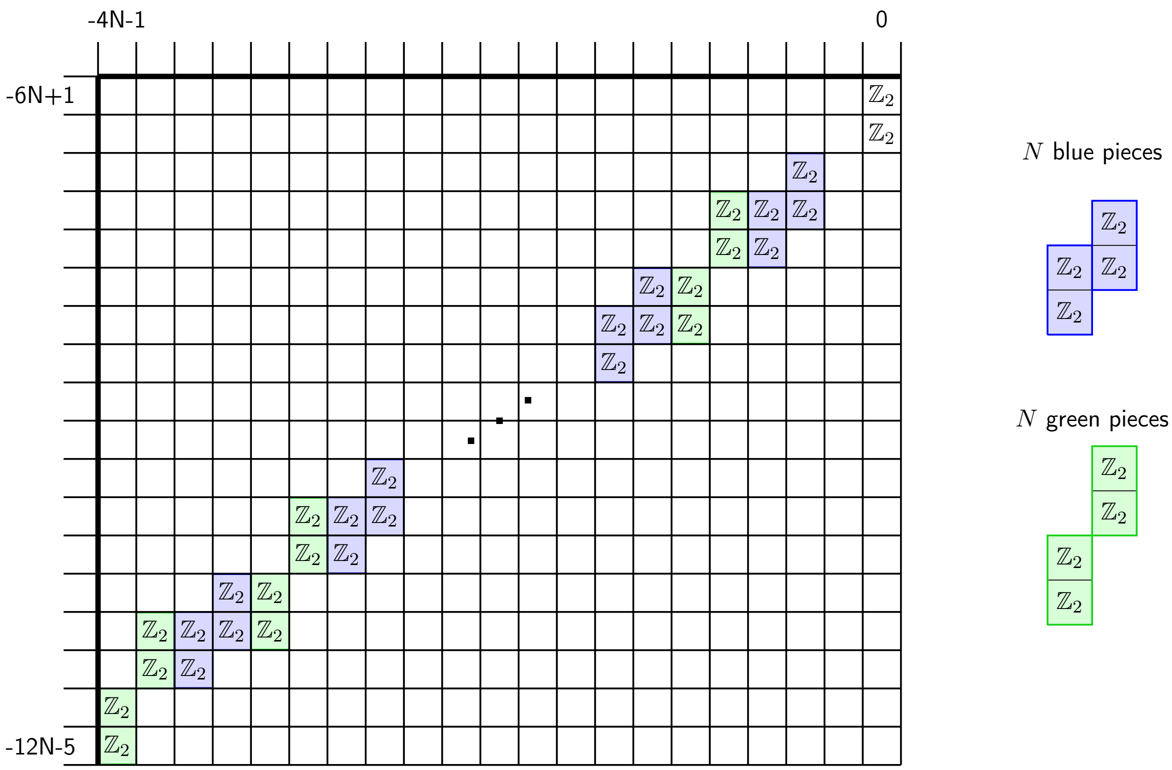


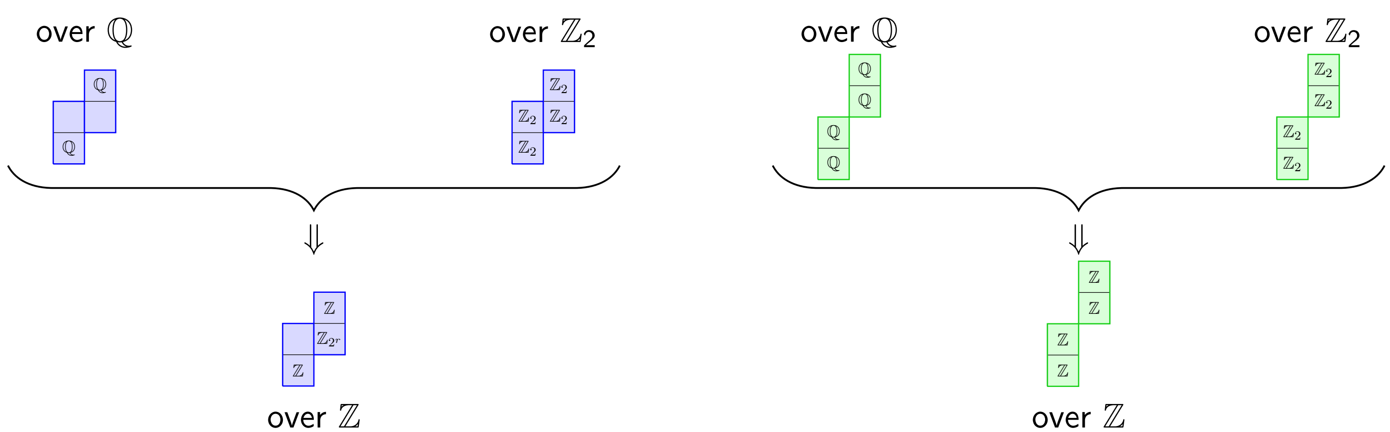
Figure: Kh($T_{3,3N+1}, \mathbb{Z}_2$) as computed by Benheddi, 2017

The blue pieces and the green pieces

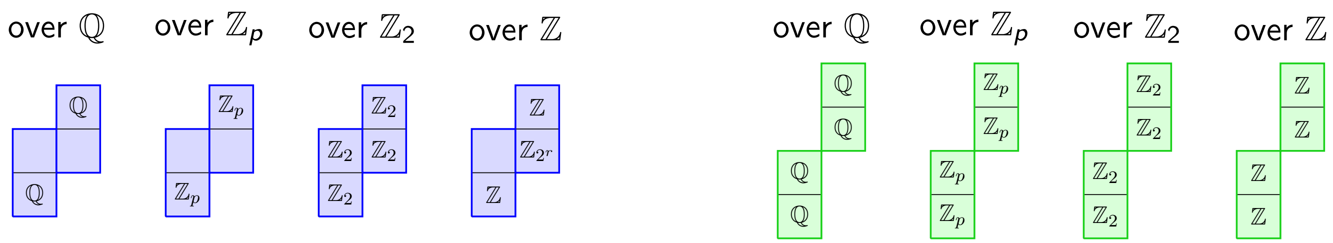
The universal coefficient theorem tells us

$$\mathrm{Kh}^{i,j}(L,\mathbb{Z}_2) \cong \big(\mathrm{Kh}^{i,j}(L)\otimes\mathbb{Z}_2\big)\oplus \mathrm{Tor}\big(\mathrm{Kh}^{i+1,j}(L),\mathbb{Z}_2\big)$$

$$\mathrm{Kh}^{i,j}(L,\mathbb{Q}) \cong \mathrm{Kh}^{i,j}(L)\otimes\mathbb{Q}$$



Computing $\mathrm{Kh}^{**}(T_{3,q})$



Observe:

- Torsion only occurs in the "blue pieces"
- The "blue pieces" are locally homologically thin over $\mathbb{Z}_p$ for all prime p
- There is no torsion in the first homological degree of each blue piece
- There is a rank equality in the first homological degree between $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients (p odd prime)

Applying the main theorem

Therefore each blue piece satisfies the conditions of the main theorem of this talk. The conclusion is the following:

Corollary (C., Lowrance, Sazdanović, Summers)

The torus links $T(3, q)$ have only $\mathbb{Z}_2$-torsion in Khovanov homology.

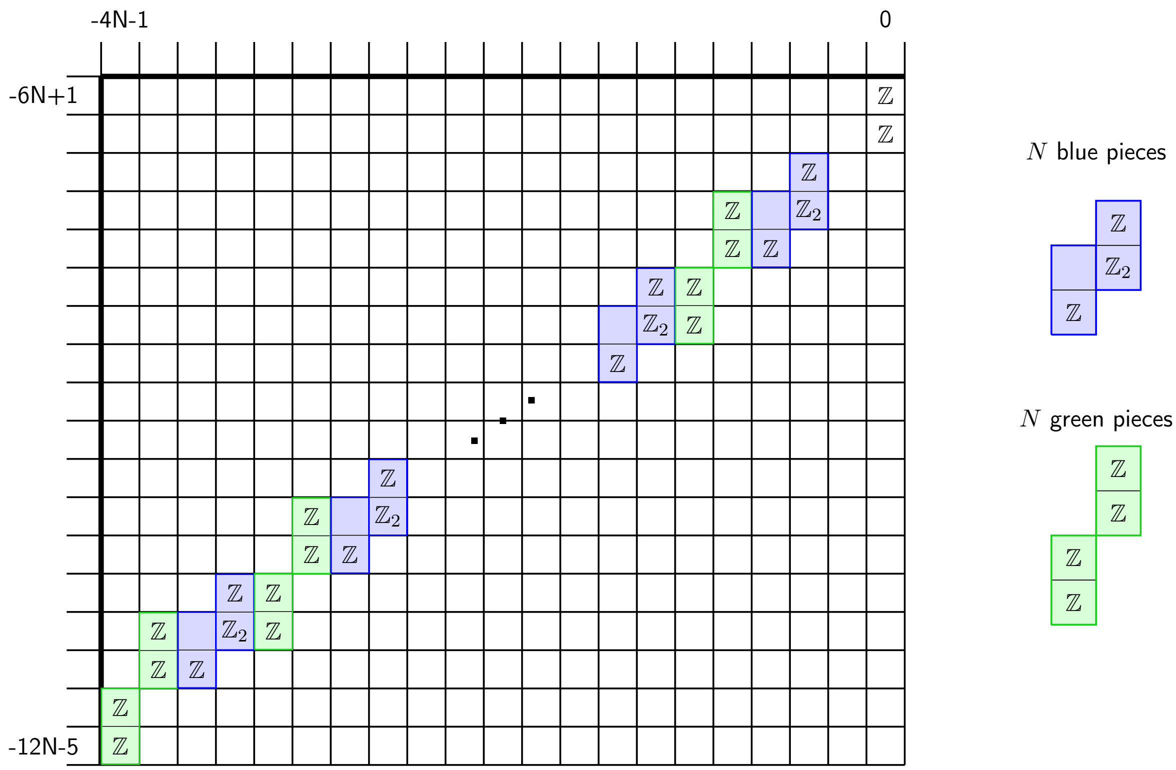


Figure: $\mathrm{Kh}^{**}(T_{3,3N+1})$ as determined by the previous theorem

Thank you!