Thin Posets, Homology Theories, and Categorification

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January 22, 2019

Categorification

Categorification is the idea of finding category theoretic analogues of set theoretic or algebraic structures:

categorification →

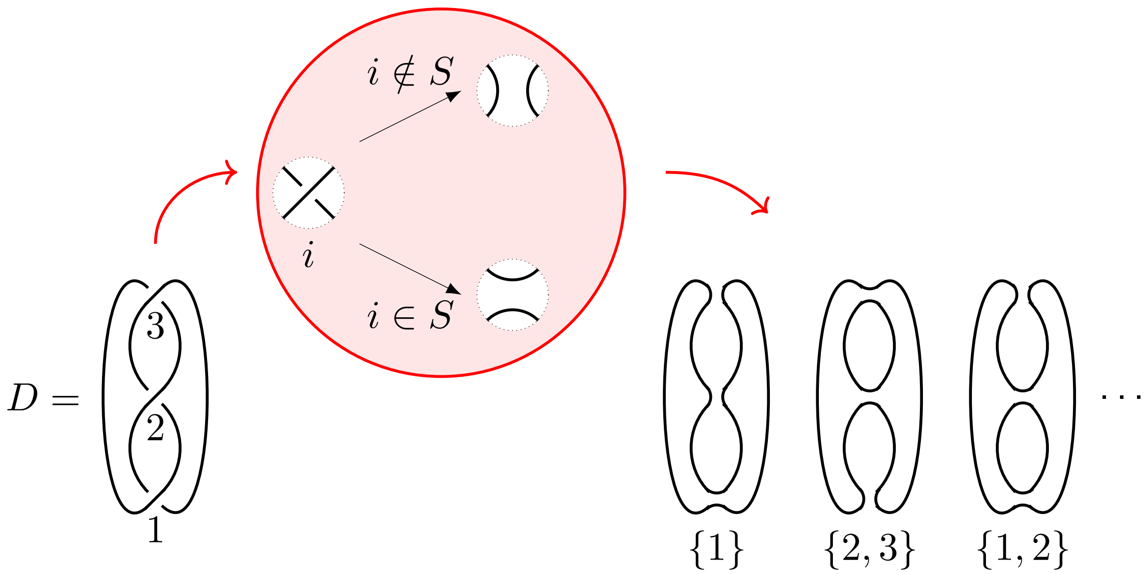
sets	categories
elements	objects
functions	functors
equations between elements	isomorphisms between objects

← decategorification

Decategorification is the reverse process (forgetting the extra structure)

An example from knot theory: Khovanov homology

- D a knot diagram with crossings $X = \{1, \ldots, n\}$
- Each $S \in 2^X$ encodes a *resolution* of D



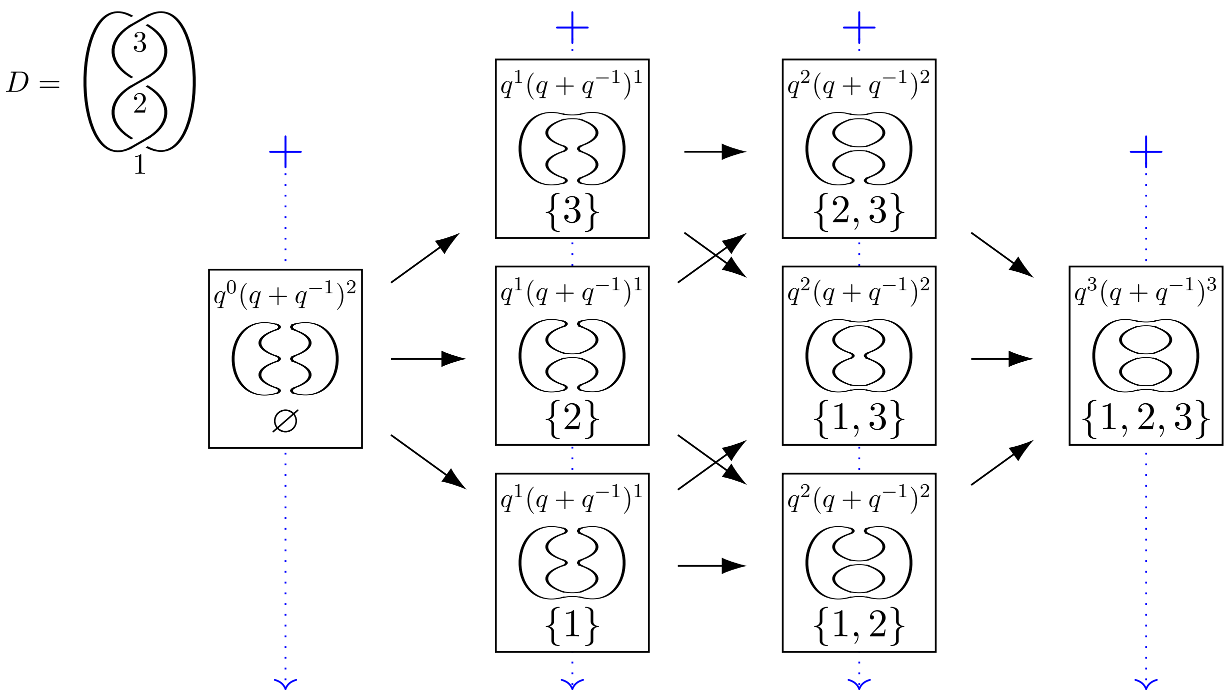
The Jones polynomial

The *Jones polynomial* (up to rescaling) has a "state sum formula":

$$J(D) = \sum_{S \in 2^X} (-1)^{|S|} q^{|S|} (q + q^{-1})^{j(S)}$$

where $j(S)$ is the number of disjoint circles in the resolution corresponding to S

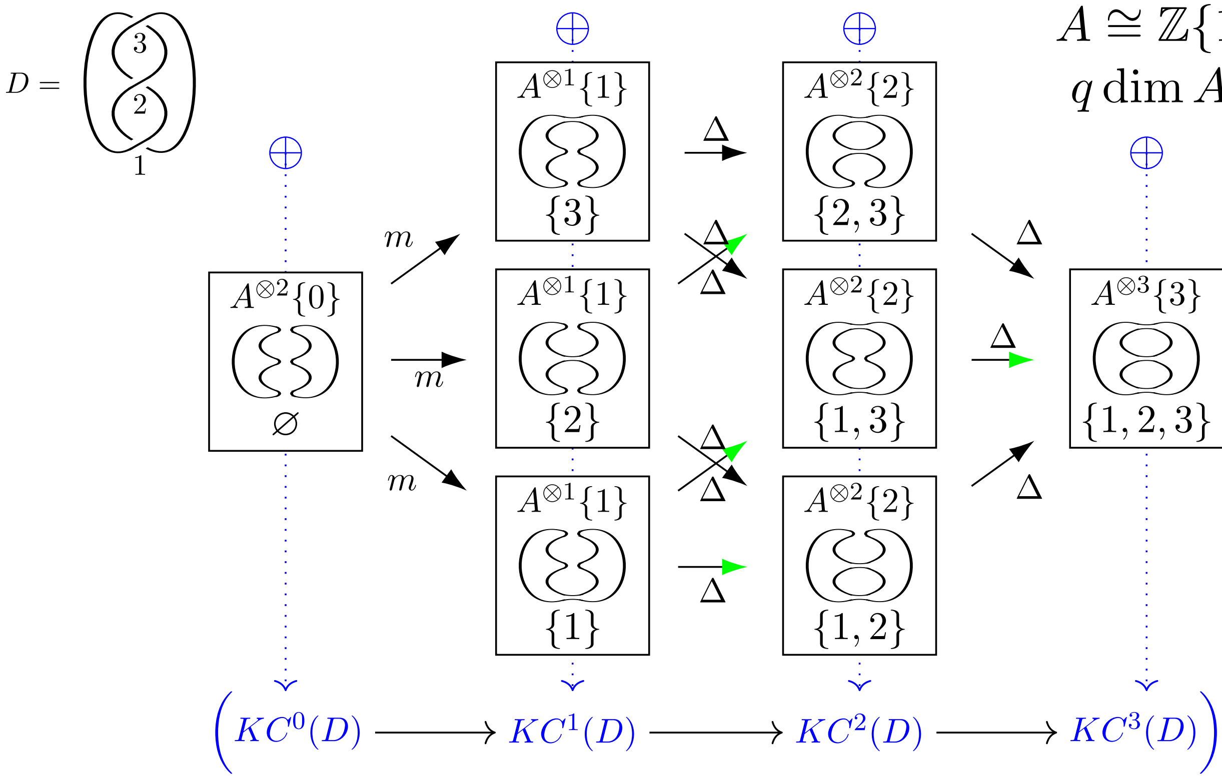
Computing the Jones Polynomial



$$J(D) = +q^0(q+q^{-1})^2 \quad -3q^1(q+q^{-1})^1 \quad +3q^2(q+q^{-1})^2 \quad -q^3(q+q^{-1})^3$$

The Khovanov 'Cube' Construction

$$A \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{-1\}$$
$$q\dim A = q + q^{-1}$$



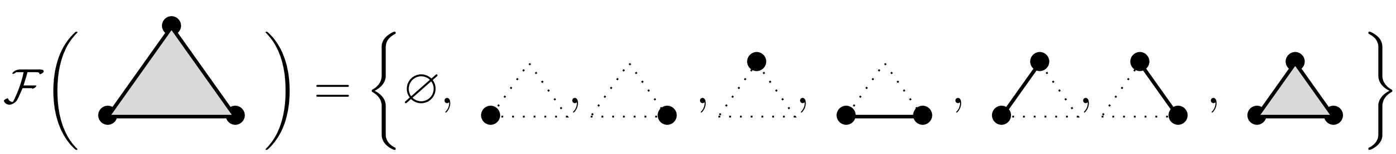
$$\chi_q(KH(D)) = \chi_q(KC(D)) = J(D)$$

Posets and Hasse Diagrams

- A **partially ordered set (poset)** $(P, \leq)$ is a set P with a reflexive, antisymmetric, and transitive relation $\leq$.
- When $x \leq y$ and $x \neq y$, we write $x < y$.
- A **cover relation** in $(P, \leq)$ is a pair $x, y \in P$ with $x < y$ such that there is no z with $x < z < y$. Write $x \lessdot y$.
- A poset is **ranked** if there is a function $\mathrm{rk} : P \to \mathbb{N}$ such that $x \lessdot y \implies \mathrm{rk}(y) = \mathrm{rk}(x) + 1$

Examples of Posets

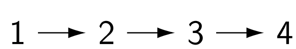
1. (chains) The set $[n] = \{1, 2, \ldots, n\}$ with the usual relation $\leq$. We have $1 \lessdot 2 \lessdot 3$ and so on. $[n]$ is ranked with $\mathrm{rk}(x) = x$.
2. (Boolean lattices) Given a set S, the collection of subsets 2^S of S is a poset with $T_1 \leq T_2$ if T_1 is contained in T_2 (usually denoted $\subseteq$). Given subsets $T_1 \subseteq T_2$, we have $T_1 \lessdot T_2$ iff $|T_2| = |T_1| + 1$. Thus 2^S is ranked by cardinality.
3. (face posets of polytopes) The set of faces $\mathcal{F}(A)$ of a polytope A is partially ordered by containment. Given faces $F_1 \subseteq F_2$, we have $F_1 \lessdot F_2$ iff $\dim F_2 = \dim F_1 + 1$. Thus face posets are ranked by dimension.



Hasse Diagrams

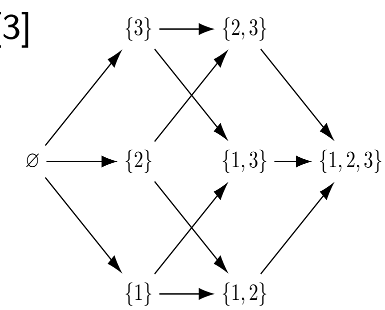
The **Hasse diagram** of a finite poset $(P, \leq)$ is a directed graph with a node for each $x \in P$ and a directed edge from x to y (drawn left to right) iff $x \lessdot y$.

E.g. $[4]$

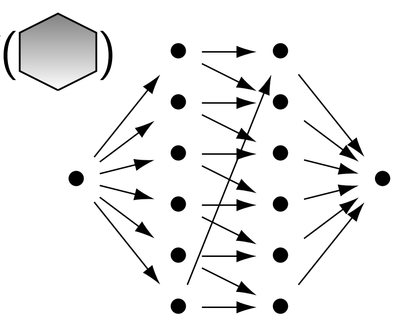


Chains

$2^{[3]}$



Boolean lattices

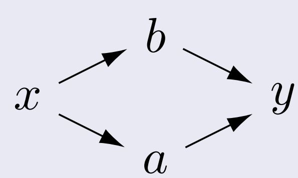
$\mathcal{F}($  $)$

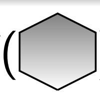
Face posets of polytopes

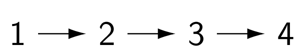
Thin Posets

Definition

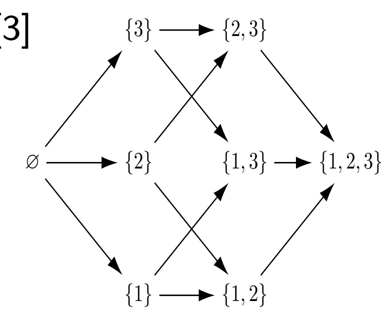
A ranked poset is **thin** if every nonempty interval $[x, y]$ with $\mathrm{rk}(y) = \mathrm{rk}(x) + 2$ is a diamond:



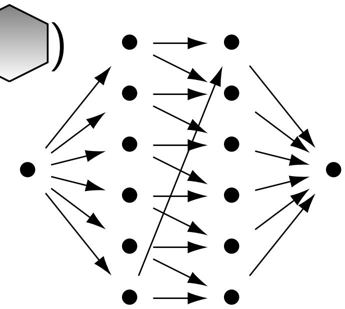
E.g. $[4]$ $2^{[3]}$ $\mathcal{F}($$)$



Chains
(not thin)



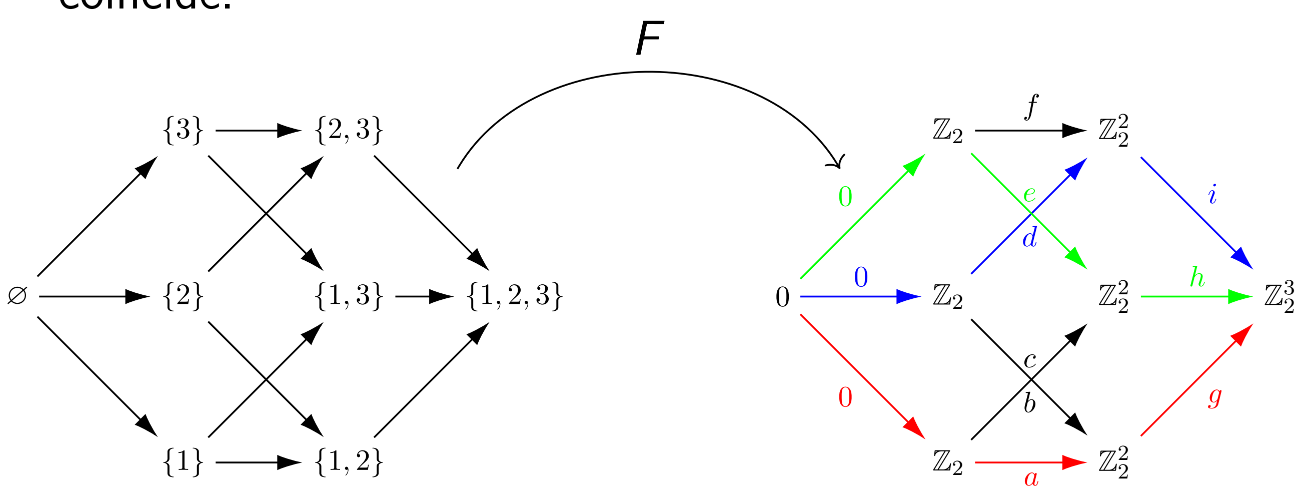
Boolean lattices
(thin)



Face posets of polytopes
(thin)

Posets as Categories

- Any poset $(P, \leq)$ can be thought of as a category: with objects P and a unique morphism from x to y iff $x \leq y$.
- A functor on a poset is then a labeling of nodes and edges of the Hasse diagram by objects and morphisms so that compositions along any two co-initial, co-terminal paths coincide.



Functors on Thin Posets Yield Homology Theories

Let P be a thin poset, $\mathcal{A}$ an abelian category, and

$$\phi : \{\text{edges in Hasse diagram}\} \to \{+1, -1\}$$

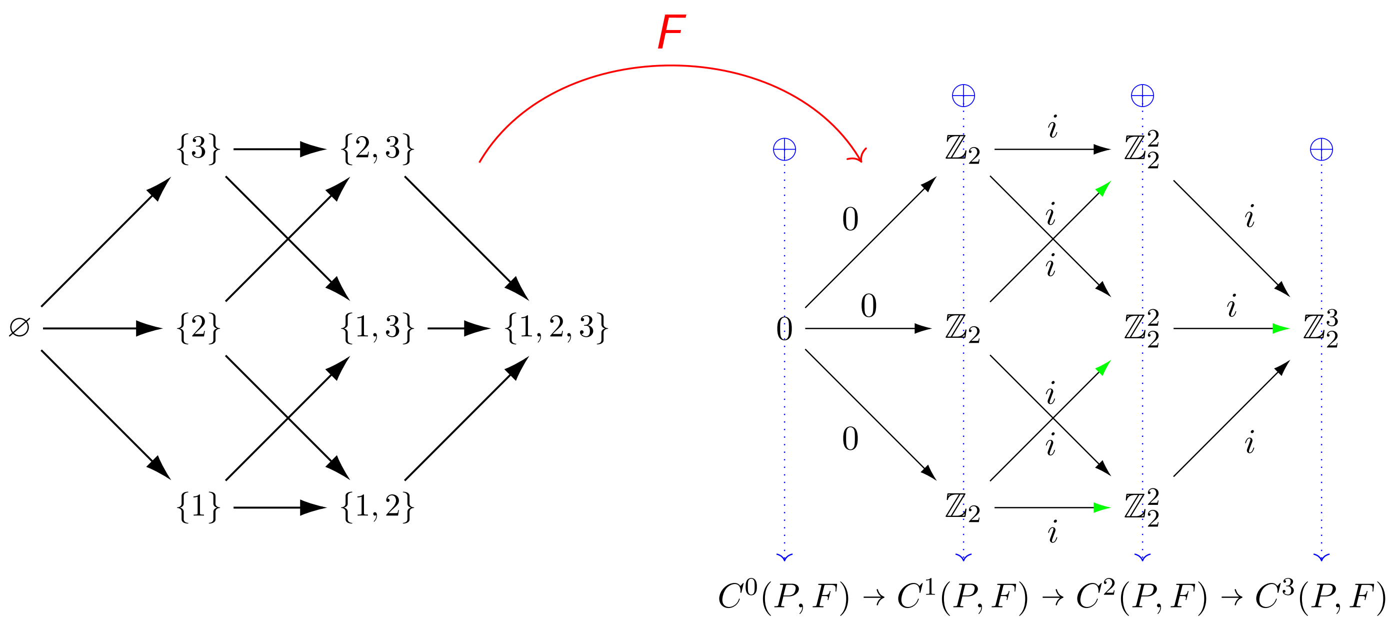
an edge coloring making diamonds anticommute.
Given a functor $F : P \to \mathcal{A}$, define a chain complex $C^*(P, F)$ by

$$C^k(P, F) = \bigoplus_{\mathrm{rk}(x)=k} F(x)$$

$$d^k : C^k(P, F) \to C^{k+1}(P, F) \qquad d^k = \sum_{\substack{x \lessdot y \\ \mathrm{rk}(x)=k}} \phi(x \lessdot y) F(x \lessdot y)$$

Since F commutes on diamonds, it follows that $d^2 = 0$. Denote the homology by $H(P, F)$.

Thin Poset Homology Pictorially



Thin poset homology and categorification

- Suppose we are interested in categorifying a ring element $g \in R$, with a formula

$$g = \sum_{x \in P} (-1)^{\mathrm{rk}(x)} f(x)$$

  where P is a thin poset, $f : P \to R$.
- Suppose that the monoidal abelian category $\mathcal{C}_R$ categorifies R in the sense that

$$K_0(\mathcal{C}_R) \cong R.$$

- If one can construct a functor $F : P \to \mathcal{C}_R$ with $[F(x)] = f(x)$ for all $x \in P$, then $H(P, F)$ categorifies g

$$\sum_{i \in \mathbb{Z}} (-1)^i [H^i(P, F)] = g$$

Vandermonde determinants

Given $\vec{s} \in \mathbb{Z}_+^n$, the corresponding **generalized Vandermonde determinant** is:

$$V_{\vec{s}}(\vec{x}) = \begin{vmatrix} x_1^{s_1} & x_1^{s_2} & \cdots & x_1^{s_n} \\ x_2^{s_1} & x_2^{s_2} & \cdots & x_2^{s_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{s_1} & x_n^{s_2} & \cdots & x_n^{s_n} \end{vmatrix} = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \dots x_n^{s_{\pi(n)}}$$

- S_n has a thin partial order (Bruhat order)
- The Bruhat order is ranked by $\mathrm{inv}(\pi)$

Categorifying the Vandermonde determinant

- Given a link diagram L with n crossings, we will construct a functor
$$F_L : S_n \to \mathcal{A}$$
from the Bruhat order on S_n to an abelian category $\mathcal{A}$ such that $[F_L(\pi)] = x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \dots x_n^{s_{\pi(n)}}$ in the Grothendieck group $K_0(\mathcal{A})$, where s_i is the number of circles in the resolution of L corresponding to $\{1, 2, \dots, i\} \subseteq [n]$.
- Thus by the previous construction, $H(S_n, F_L)$ categorifies the generalized Vandermonde determinant
$$V_L(\vec{x}) = \det(x_i^{s_j}).$$

The category of colored cobordisms: $\mathbf{Cob}_2^n$

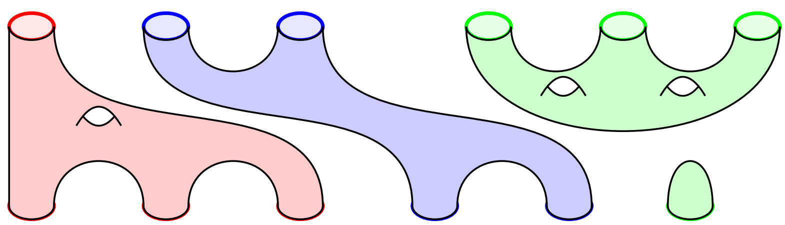
Objects: $[n]$-colored closed 1-manifolds

Morphisms: color preserving cobordisms

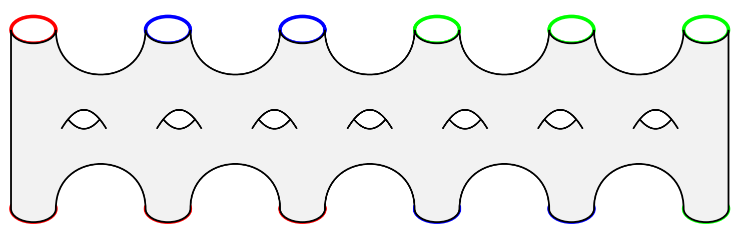
Let $[n] = \{1, 2, ..., n\}$. The category $\mathbf{Cob}_2^n$ has

- Objects: closed oriented 1-manifolds with each connected component given a color from $[n]$
- Morphisms: 2-dimensional oriented manifolds for which each connected component has monochromatic boundary

For example, let $M =$ ○○○○○○ and $N =$ ○○○○○○



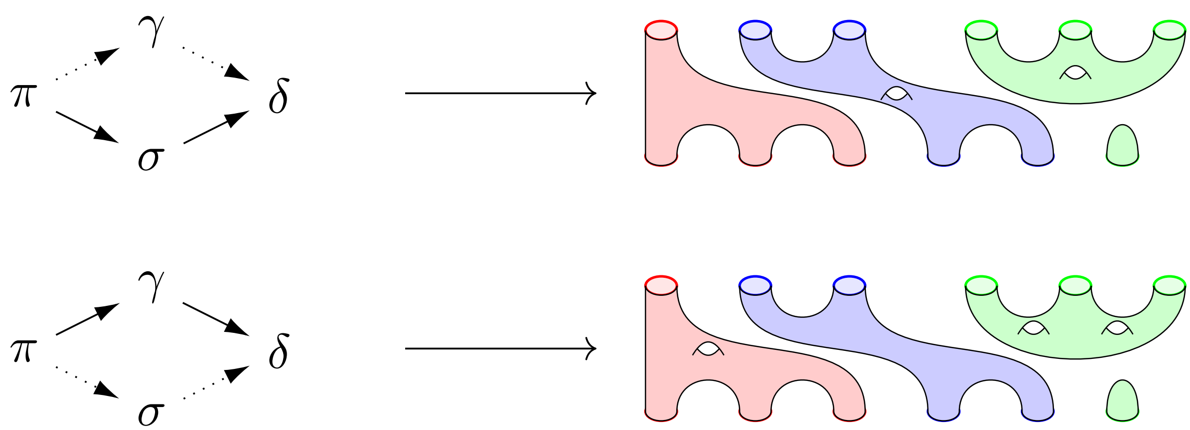
is a colored cobordism from M to N, but not



Method for defining $F_L : S_n \to \mathcal{A}$

We will define F_L as follows:

- Define a 'functor' G_L from S_n to $\mathbf{Cob}_2^n$
- Composition law holds only up to 'stabilization', i.e. possibly up to connect summing with an appropriate number of tori



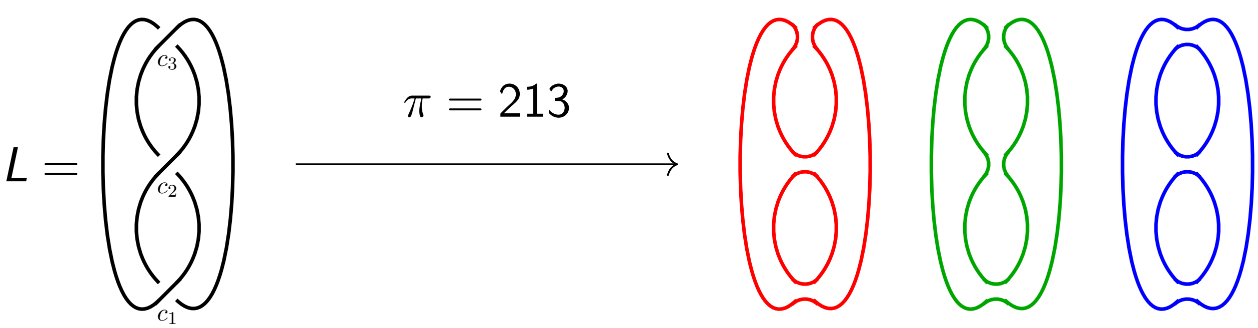
- Post compose with a functor $Z_L : \mathbf{Cob}_2^n \to \mathcal{A}$ which acts invariantly under stabilization

Define $G_L : S_n \to \mathbf{Cob}_2^n$ on objects

L a link diagram with crossings $c_1, ..., c_n$. For $\pi \in S_n$ define

$$F_L(\pi) = L_1^\pi \amalg L_2^\pi \amalg ... \amalg L_n^\pi \in \text{Ob Cob}_2^n$$

where L_i^π the resolution of L corresponding to $\{1, 2, \ldots, \pi(i)\}$, and all components of L_i^π are colored i.



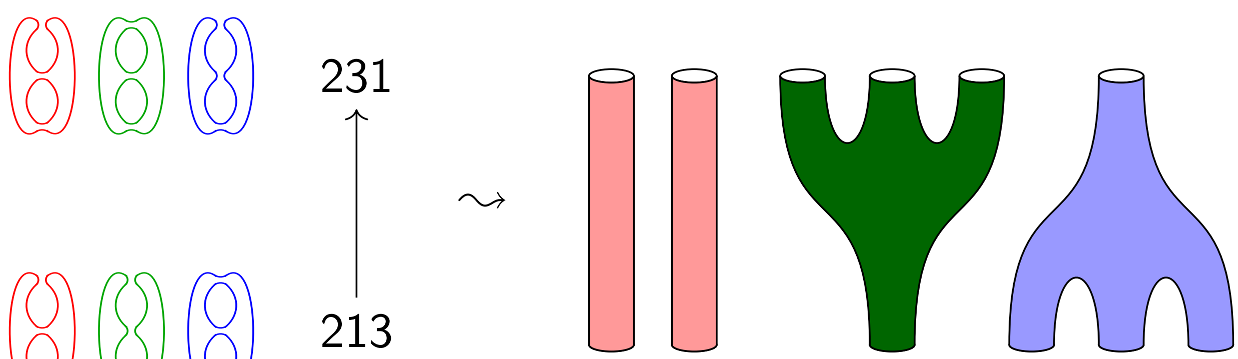
$$F_L(\pi) = L_1^\pi \amalg L_2^\pi \amalg L_3^\pi$$

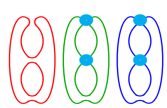
Define $G_L : S_n \to \mathbf{Cob}_2^n$ on morphisms

If $\pi \lessdot \sigma$ then $K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \mathrm{Ob}\ \mathrm{Cob}_2^n$ and

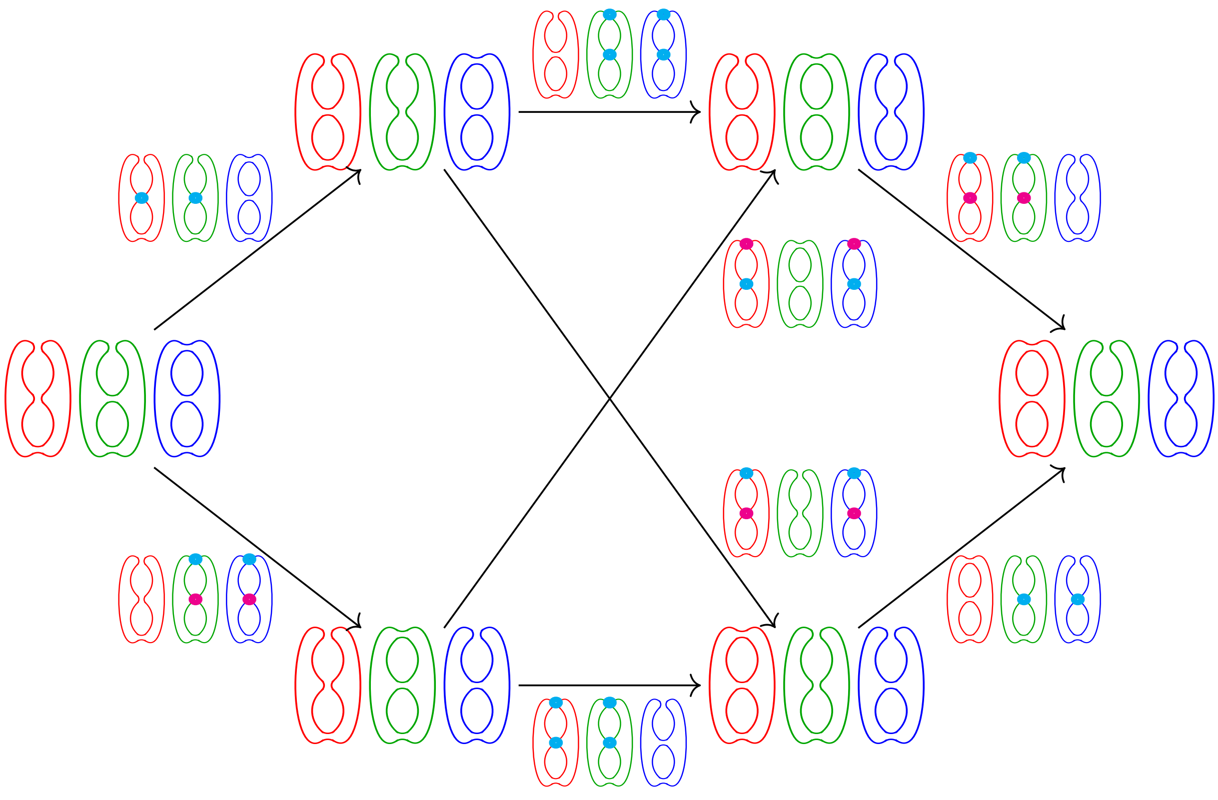
$$K^\sigma = K_1^\sigma \amalg K_2^\sigma \amalg \ldots \amalg K_n^\sigma \in \mathrm{Ob}\ \mathrm{Cob}_2^n$$

differ at exactly two colors. Use connected genus 0 cobordisms on the colored pieces which differ, and identity (cylinders) on pieces which do not change



notation: 

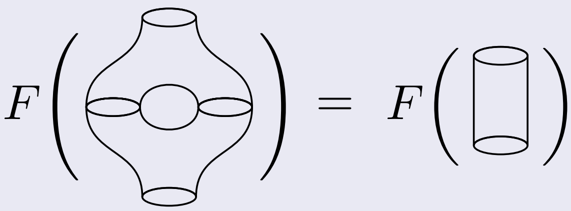
We have defined a 'functor' $G_D : S_n \to \mathbf{Cob}_2^n$



2D special colored TQFTs

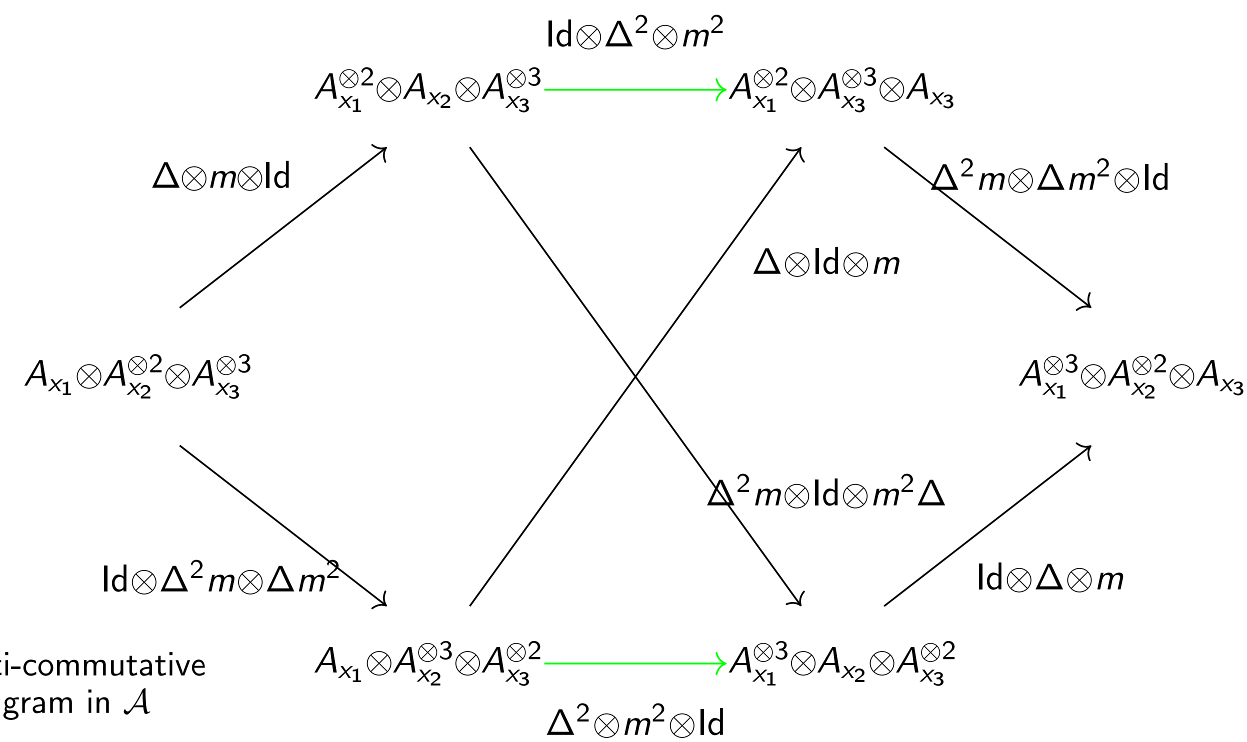
Definition

- A **2D TQFT** is a symmetric monoidal functor $Z : \mathbf{Cob}_2^1 \to \mathcal{A}$ where $\mathcal{A}$ is symmetric monoidal abelian
- A 2D TQFT F is **special** if the following condition holds:



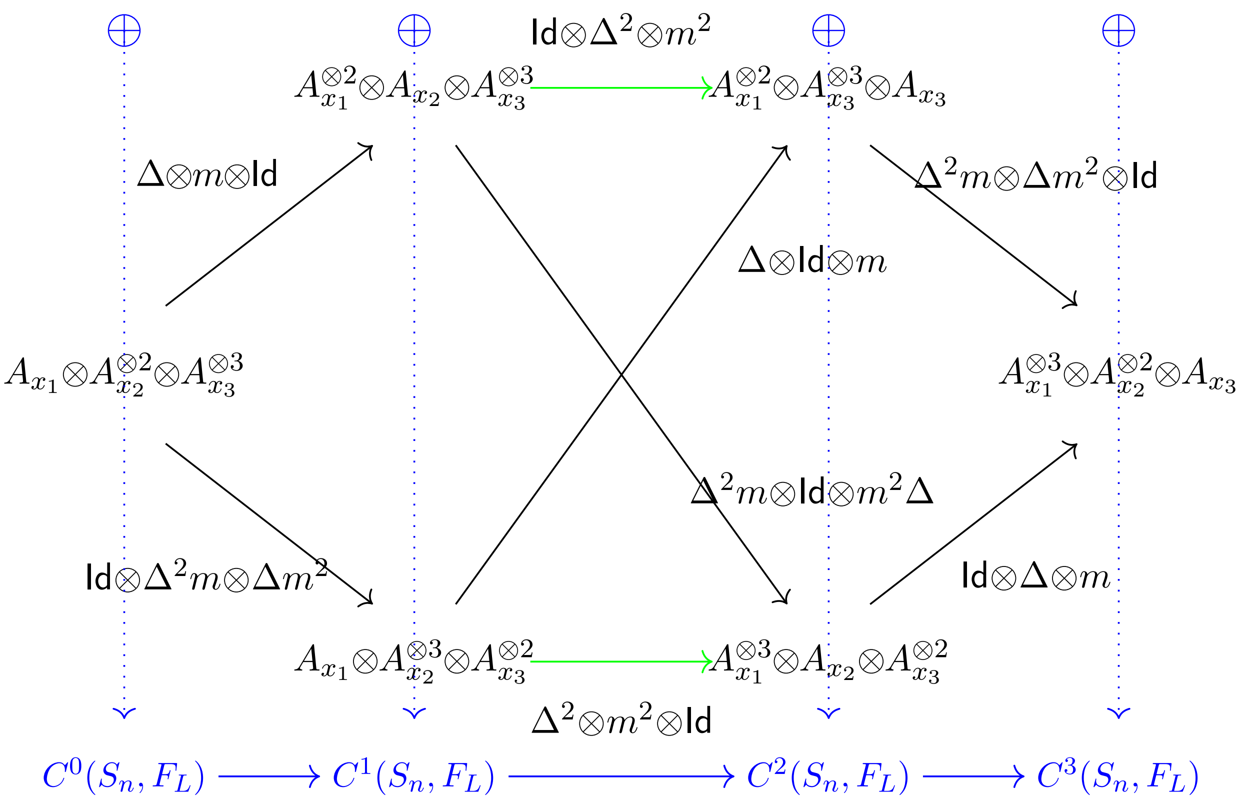
- Note that $\mathbf{Cob}_2^n \cong \mathbf{Cob}_2^1 \times \cdots \times \mathbf{Cob}_2^1$
- A **special colored TQFT** is a monoidal functor $F : \mathbf{Cob}_2^n \to \mathcal{A}$ which restricts to a special TQFT on each color (each copy of $\mathbf{Cob}_2^1$).

Apply a Special Colored TQFT



An anti-commutative diagram in $\mathcal{A}$

Form a chain complex



A Categorification of the Vandermonde determinant

Theorem (C., 2016)

Let Z be a special colored TQFT, $Z : \mathbf{Cob}_2^n \to \mathcal{A}$, let $F_L = Z \circ G_L$, and let x_i denote $[Z(\bigcirc_i)] \in K_0(\mathcal{A})$. For any link diagram L,

$$\sum_{i\in\mathbb{Z}} (-1)^i [H^i(S_n, F_L)] = V_L(\vec{x})$$

Thank you!