Thin Posets and Diamond Transitivity

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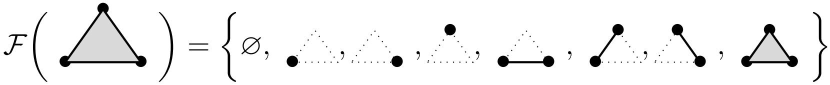
Partially Ordered Sets (Posets)

- A **partially ordered set (poset)** $(P, \leq)$ is a set P with a reflexive, antisymmetric, and transitive relation $\leq$.
- When $x \leq y$ and $x \neq y$, we write $x < y$.
- A **cover relation** in $(P, \leq)$ is a pair $x, y \in P$ with $x < y$ such that there is no z with $x < z < y$. Write $x \lessdot y$.
- A poset is **ranked** if there is a function $\mathrm{rk} : P \to \mathbb{N}$ such that $x \lessdot y \implies \mathrm{rk}(y) = \mathrm{rk}(x) + 1$

Examples of Posets

1. (chains) The set $[n] = \{1, 2, \ldots, n\}$ with the usual relation $\leq$. We have $1 \lessdot 2 \lessdot 3$ and so on. $[n]$ is ranked with $\mathrm{rk}(x) = x$.
2. (Boolean lattices) Given a set S, the collection of subsets 2^S of S is a poset with $T_1 \leq T_2$ if T_1 is contained in T_2 (usually denoted $\subseteq$). Given subsets $T_1 \subseteq T_2$, we have $T_1 \lessdot T_2$ iff $|T_2| = |T_1| + 1$. Thus 2^S is ranked by cardinality.
3. (face posets of polytopes) The set of faces $\mathcal{F}(A)$ of a polytope A is partially ordered by containment. Given faces $F_1 \subseteq F_2$, we have $F_1 \lessdot F_2$ iff $\dim F_2 = \dim F_1 + 1$. Thus face posets are ranked by dimension.



Hasse Diagrams

The **Hasse diagram** of a finite poset $(P, \leq)$ is a directed graph with a node for each $x \in P$ and a directed edge from x to y (drawn left to right) iff $x \lessdot y$.

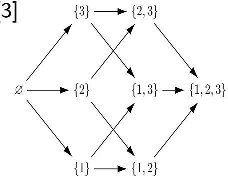
E.g.

$[4]$

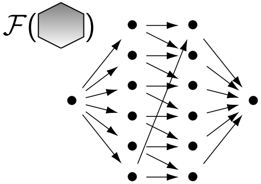
$$1 \to 2 \to 3 \to 4$$

Chains

$2^{[3]}$



Boolean lattices

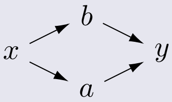


Face posets of polytopes

Thin Posets

Definition

A ranked poset is **thin** if every nonempty interval $[x, y]$ with $\mathrm{rk}(y) = \mathrm{rk}(x) + 2$ is a diamond:

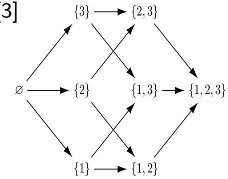


E.g. $[4]$

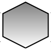
$$1 \to 2 \to 3 \to 4$$

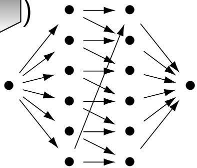
Chains
(not thin)

$2^{[3]}$



Boolean lattices
(thin)

$\mathcal{F}($$)$



Face posets of polytopes
(thin)

Categories and Functors

A **category** consists of

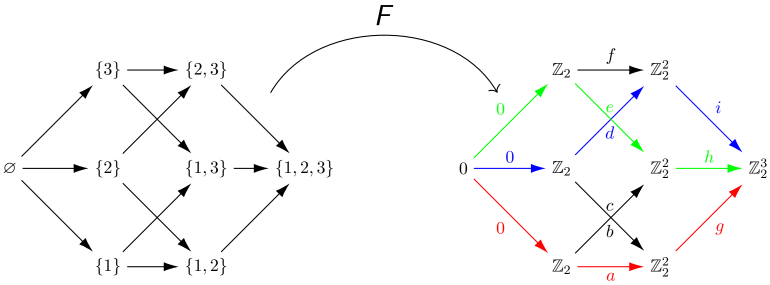
- a class of **objects**
- for any objects A, B, a class of **morphisms** $\mathrm{Mor}(A, B)$
- a way to '**compose**' morphisms, and identity morphisms w.r.t. this composition

Category	Objects	Morphisms
$\Bbbk$-**Vect**	$\Bbbk$-vector spaces	$\Bbbk$-linear maps
Top	topological spaces	continuous maps
$\mathbf{Cob}_n$	closed n-manifolds	compact $n+1$-manifolds w/ bdry

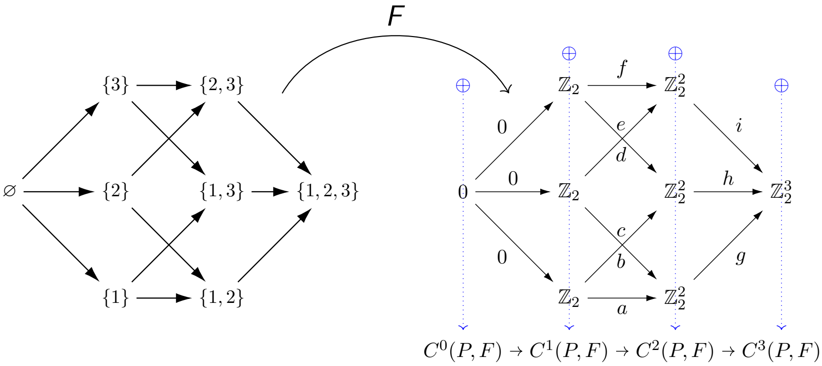
A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ sends objects in $\mathcal{C}$ to objects in $\mathcal{D}$, morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$, and preserves compositions and identity morphisms.

Posets as Categories

- Any poset $(P, \leq)$ can be thought of as a category: with objects P and a unique morphism from x to y iff $x \leq y$.
- A functor on a poset is then a labeling of nodes and edges of the Hasse diagram by objects and morphisms so that compositions along any two co-initial, co-terminal paths coincide.



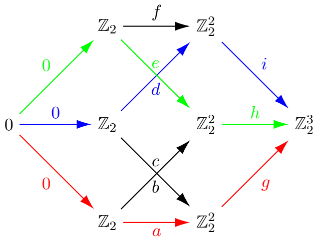
Functors on Thin Posets Yield Homology Theories



E.g. Khovanov homology, simplicial homology, Morse homology, chromatic homology, etc...

Constructing Functors on Thin Posets

To construct a functor on a poset, we begin by labeling nodes by objects of a category and directed edges by morphisms. We then have to check whether compositions along co-initial, co-terminal paths coincide.

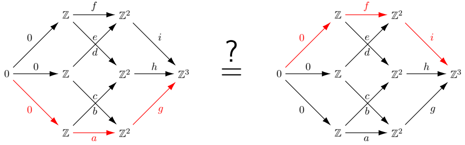


Question

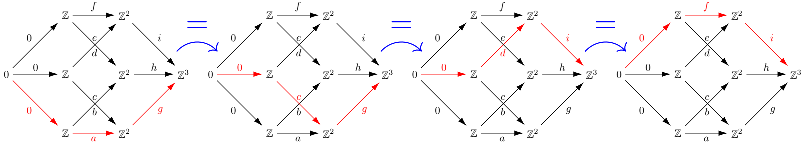
Let P be a thin poset. To construct a functor on P, is it enough for morphisms to commute just on diamonds?

Constructing Functors on Thin Posets

I.e. Suppose we have a labeling of Hasse diagram and we know morphisms commute on diamonds. Can we use this to show morphisms commute on longer paths?



Idea: commute along one diamond at a time



Remark: this is a purely combinatorial question.

Diamond Transitivity

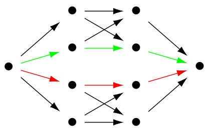
Definition

Say a thin poset P is **diamond transitive** if the answer to the previous question is yes.

Question

Are all thin posets diamond transitive?

No! Counter example:



i.e. pinch two thin posets together of rk ≥ 3

Question

Is this the only type of obstruction?

Question: Which thin posets are diamond transitive?

Theorem (C., Hollering, Lacina 2018)

The following types of posets are diamond transitive:

1. *Face posets of simplicial complexes (in particular Boolean lattices)*
2. *Face posets of polytopal complexes (in particular face posets of polytopes)*
3. *Thin shellable posets*

Question: Which thin posets are diamond transitive?

Conjecture (C., Hollering, Lacina)

Let P be a thin poset with a unique minimal element $\hat{0}$. Then P is diamond transitive if and only if P is isomorphic to the face poset of a regular CW complex.

Thank you!