A Categorification of the Laguerre Polynomials

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Abstract

We continue the work of Khovanov and Sazdanovic in [1] categorifying certain families of orthogonal polynomials. Similar to their categorification of Hermite and Chebyshev polynomials, in this paper Laguerre polynomials are lifted to modules over a diagrammatic algebra. Expressions for Laguerre polynomials in terms of the monomial basis become projective resolutions of modules which correspond to Laguerre polynomials, and the inverse relation becomes a filtration of modules.

1 Introduction

In [1], and a pair of unpublished sequels, Khovanov and Sazdanović provide a framework for categorifying orthogonal polynomials. Given a family $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}[x]$ of polynomials orthogonal with respect to the inner product $(\cdot,\cdot)$. The goal is to realize $\mathbb{Z}[x]$ as the Grothendieck group of some category. Monomials x^n should lift to indecomposable objects P_n in such a way that $(x^n, x^m) = \dim \mathrm{Hom}(P_n, P_m)$. This is accomplished by looking at a category of modules over a *diagrammatic algebra* A, that is, a $\Bbbk$-algebra with a $\Bbbk$-basis of planar (n,m)-diagrams, where by a planar (n,m)-diagram, we will mean (somewhat vaguely) a subset of $[0,1]\times\mathbb{R}$ consisting of n points on the line $x=0$, m points on the line $x=1$, and some collection of curves attached to the points. Multiplication in a diagrammatic algebra is done by concatenating compatible diagrams and setting incompatible products to 0. Exactly which diagrams are in the basis will be specified in each of the different choices for orthogonal polynomials. Generally, diagrammatics should be chosen so that there is a unique idempotent (n,n)-diagram, which we can denote 1_n, since in this case we can define $P_n = A1_n$ and this gives a family of projective indecomposables for which $\mathrm{Hom}(P_n, P_m)$ has a basis of (n,m)-diagrams. Thus under this setup, we should choose diagrammatics for which the number of (n,m)-diagrams is equal to (x^n, x^m).

Depending on the choice of polynomials/diagrammatics, we will need to choose which modules to include in our category. Under certain conditions (for instance if A is Noetherian), one can show that any finitely generated projective A-module decomposes uniquely as a direct sum of P_n's and thus the Grothendieck group of the category of finitely generated projective A-modules is isomorphic to $\mathbb{Z}[x]$ via $[P_n]\mapsto x^n$. One can additionally define a monoidal structure on this category to get an isomorphism of rings.

To categorify the polynomials $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}[x]$ we search for modules M_n admitting some relation with the modules P_k (for example, a projective resolution or filtration) which descends to a closed formula for f_n in terms of monomials x^k in the Grothendieck group $K_0(A\text{-pmod}) \cong \mathbb{Z}[x]$. This could be accomplished in a trivial way by perhaps taking appropriate direct sums of the P_k, but instead we search for modules M_n with a geometric relation to the P_k's via diagrammatics.

AC: state main results here

In Section 2 we give a review of Khovanov and Sazdanovic's slarc algebra which leads to a categorification of $\mathbb{Z}[x]$ and polynomials $(x-1)^n$. Section 3 describes the unpublished sequel to [1] where they categorify Hermite and Chebyshev polynomials. In Section 4 we show how to modify the general setup given by Khovanov and Sazdanovic in a way which can be used to categorify exponential polynomials (polynomials in the variable $\frac{x^n}{n!}$, and in particular, Laguerre polynomials. In $\mathbb{Z}[x]$ the product $x^n x^m = x^{n+m}$ is easily categorified (see sections 2 and 3) but in the exponential polynomial ring, the product $\frac{x^n}{n!}\frac{x^m}{m!} = \binom{n+m}{n}\frac{x^{n+m}}{(n+m)!}$ is more complicated. We show a way to categorify this product in Section 6.

2 Slarcs and a Categorification of the Polynomial Ring $\mathbb{Z}[x]$

In this section, we give an outline of Khovanov and Sazdanovic's categorification [1] of $\mathbb{Z}[x]$. The first ingredient in their categorification is a diagrammatic algebra A^-.

Definition 2.1. *An (n, m)-slarc diagram is an isotopy class of 1-dimensional submanifolds of $[0,1] \times \mathbb{R}$ with*

1. *n boundary points at $x = 0$ and m boundary points at $x = 1$,*
2. *well defined slope at each point,*
3. *every connected component has at least one boundary point on $\{0,1\} \times \mathbb{R}$, and all boundary points are on $\{0,1\} \times \mathbb{R}$.*

Connected components with two boundary points are called long arcs (or larcs) and connected components with only one boundary point are called short arcs (or sarcs). The number of larcs in a diagram is called the width.

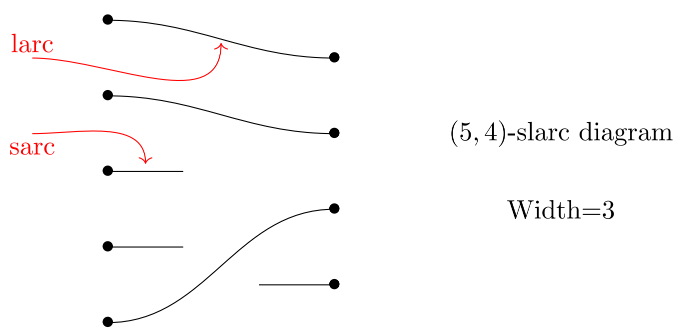


Figure 1: An example of a $(5, 4)$-slarc diagram.

See figure 1 for an example of a slarc diagram. Let ${}_nB_m$ denote the set of all (n, m)-slarc diagrams, and let $B = \coprod_{n,m\geq 0} {}_nB_m$. Elements of B will typically be denoted by lower case letters $a, b, c, d, e, ...$ early in the alphabet. Let A^- denote the $\Bbbk$-vector space with basis B. Define a product on A^- by letting $a \cdot b$ be the horizontal concatenation of a and b if the number of right endpoints of a is equal to the number of left endpoints of b, or zero otherwise. Call A^- the *slarc algebra*. The diagram $1_n \in {}_nB_n$ consisting of n larcs is an idempotent in A^- and the family $\{1_n \mid n \in \mathbb{N}\}$ is mutually orthogonal, endowing A^- with the structure of an idempotented algebra: $A^- = \bigoplus_{n,m\geq 0} 1_m A^- 1_n$. The idempotents A^- serve as a substitute for a multiplicative identity element. Consider the category A^--pmod of projective finitely generated left A^--modules. We will be interested in the indecomposable projective modules $P_n = A^- 1_n$. One can show that $\mathrm{Hom}(P_n, P_m)$ is generated by slarc diagrams with n left and m right endpoints, and thus

$$\dim \mathrm{Hom}(P_n, P_m) = \binom{n+m}{n}. \tag{1}$$

We will also need the *standard module* M_n, defined as the quotient of P_n by the submodule consisting of all diagrams in P_n with at least one right sarc.

Definition 2.2. *The split Grothendieck group of an abelian category $\mathcal{A}$ is the abelian group $K_0(\mathcal{A})$ generated by symbols $[M]$ for objects $M \in \mathcal{A}$ modulo the relations $[A] = [B] + [C]$ whenever $A \cong B \oplus C$ in $\mathcal{A}$.*

The following is proved in [1]:

Theorem 2.3. *For any field $\Bbbk$,*

1. *Any object P in A^--pmod is isomorphic to a direct sum $P \cong \oplus_{n\in\mathbb{N}} P_n^{a_n}$ and the a_n are invariants of P.*

2. $K_0(A^-\text{-pmod}) \cong \mathbb{Z}[x]$ *via* $[P_n] \mapsto x^n$.

3. *Any simple* A^-*-module is isomorphic to the one dimensional module* $L_n = \Bbbk 1_n$ *on which any element of* B *other than* 1_n *acts by zero.*

4. P_n *has a filtration*
$$P_n = P_n(\leq n) \supseteq P_n(\leq n-1) \supseteq \cdots \supseteq P_n(\leq 0) = 0$$
where $P_n(\leq m)$ *is spanned by diagrams in* B_n *of width at most* m *and subsequent quotients* $P_n(\leq m)P_n(\leq m-1)$ *are isomorphic to* $\binom{n}{m}$ *copies of the standard module* M_m.

5. M_n *has a resolution*
$$0 \longrightarrow P_0^{\binom{n}{n}} \longrightarrow \ldots \longrightarrow P_{n-m}^{\binom{n}{m}} \longrightarrow \ldots \longrightarrow P_{n-1}^{\binom{n}{1}} \longrightarrow P_n^{\binom{n}{0}} \longrightarrow M_n \longrightarrow 0$$
in which each arrow consists of a matrix of diagrams, each diagram having exactly one sarc.

AC: can turn all A^- into A

In the Grothendieck group $K_0(A^-\text{-pmod})$, parts 4 and 5 in Theorem 2.3 yield relations:

$$[P_n] = \sum_{m \geq 0} \binom{n}{m} [M_m], \tag{2}$$

and

$$[M_n] = \sum_{m \geq 0} (-1)^{n+m} \binom{n}{m} [P_m]. \tag{3}$$

In terms of the isomorphism $K_0(A^-) \cong \mathbb{Z}[x]$ sending $[P_n] \mapsto x^n$, equation (3) says

$$[M_n] = \sum_{m \geq 0} (-1)^{n+m} \binom{n}{m} x^m = (x-1)^n,$$

and equation (2) gives the inverse relation

$$x^n = \sum_{m \geq 0} \binom{n}{m} (x-1)^m.$$

AC: However, in this example, there is no inner product and the polynomials $(x-1)^n$ are not orthogonal in any sense (are they?).

AC: Talk about BGG duality

AC: What about MacDonald polynomials?

3 Categorified Hermite and Chebyshev Polynomials

In this section we recall the work of Khovanov and Sazdanović in categorifying the Chebyshev and Hermite polynomials. This will help to set up some notation and get a flavor for how these diagrammatic categorifications work for orthogonal polynomials.

3.1 Chebyshev Polynomials

The Chebyshev polynomials (of the second kind) are defined on the interval $(-1, 1)$ and are orthogonal with respect to the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}dx$. We consider the rescaled version $\mathcal{U}_n(x)$ which forms an orthonormal set on the interval $(-2, 2)$ with the inner product $(f, g) = \frac{1}{2\pi}\int_{-2}^{2} f(x)g(x)\sqrt{1-4x^2}dx$. For this inner product we have $(x^n, x^m) = C_{\frac{n+m}{2}}$ if $n+m$ is even and 0 otherwise, where C_k is the k^{th} catalan

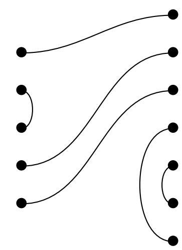


Figure 2: An example of an (n, m)-Chebyshev diagram

number. Thus (x^n, x^m) is the number of non-crossing matchings of n dots located at $x = 0$ and m dots located at $x = 1$. One finds the explicit formula

$$\mathcal{U}_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}. \tag{4}$$

Definition 3.1. *An (n, m)-Chebyshev diagram is an isotopy class of planar diagrams D in the strip $[0, 1] \times \mathbb{R} \subseteq \mathbb{R}^2$ with:*

1. *n vertices on the line $x = 0$*
2. *m vertices on the line $x = 1$*
3. *Each vertex is the endpoint of an arc in D*
4. *Arcs in D are only allowed to begin and end at one of the $n + m$ vertices in $\{0, 1\} \times \mathbb{R}$*
5. *Arcs beginning at $x = 0$ and ending at $x = 1$ (call these through arcs) must have well defined slope everywhere*
6. *Arcs beginning and ending at the same x value (call these returns) have exactly one point with undefined slope*
7. *Arcs have no self intersections*
8. *No two distinct arcs intersect*
9. *Only isotopies preserving the above conditions are allowed*

AC: need a unified language for the three diagrammatic definitions (slarc, chebyshev, hermite)

Let ${}_nB^C_m$ denote the set of all (n, m)-Chebyshev diagrams, and $B^C_m = \coprod_{n \in \mathbb{N}} {}_nB^C_m$. Let $\Bbbk$ be a field, let ${}_nA^C_m$ be the $\Bbbk$-vector space with basis ${}_nB^C_m$, and define

$$A^C = \bigoplus_{n,m \geq 0} {}_nA^C_m.$$

Endow A^C with a non-unital algebra structure by defining multiplication as follows:

- Given $x \in {}_nB^C_m$ and $y \in {}_mB^C_\ell$, define the product $xy \in {}_nB^C_\ell$ by concatenating horizontally. If the product xy defined in this way yields a diagram which does not satisfy all conditions in Definition 3.2, set $xy = 0$.
- Given $x \in {}_nB^C_m$ and $y \in {}_kB^C_\ell$ with $m \neq k$, set $xy = 0$.
- Extend to a multiplication on A^C by extending linearly from the action on the basis elements.

The algebra A^C has a family of mutually orthogonal idempotent elements $\{1_n\}_{n\in\mathbb{N}}$ where 1_n is the unique element of ${}_nB_n^C$ with no crossings. Define $P_n = A^C 1_n$. The modules $\{P_n\}_{n\in\mathbb{N}}$ are projective and locally finite dimensional (meaning $1_m P_n$ is finite dimensional for any $m \geq 0$). Let A^C-plfd denote the category of projective locally finite dimensional A^C-modules. Any object in A^C-plfd is isomorphic to a finite direct sum of P_n's and the Grothendieck group $K_0(A^C\text{-plfd})$ is free abelian with basis $\{[P_n]\}_{n\in\mathbb{N}}$, and therefore $K_0(A^C\text{-plfd}) \cong \mathbb{Z}[x]$ via the map $[P_n] \mapsto x^n$. Let $\mathcal{C}(A^C\text{-plfd})$ denote the category of chain complexes over A^C-plfd. There is a canonical isomorphism $K_0(\mathcal{C}(A^C\text{-plfd})) \cong K_0(A^C\text{-plfd})$ by sending the symbol of a complex to its Euler characteristic $[X] \mapsto \sum_{i\in\mathbb{Z}}(-1)^i[X^i]]$.

AC: maybe these should be M_n^C? Let M_n denote the quotient of P_n by the submodule spanned by diagrams with n right endpoints and at least one right return. Let $P_n(\leq w)$ denote the submodule of P_n generated by diagrams in P_n of width less than or equal to w, for $w \geq 0$. Then P_n has a filtration

$$P_n = P_n(\leq n) \supseteq P_n(\leq n-2) \supseteq \cdots \supseteq P_n(\leq n-2k) \supseteq \cdots \supseteq P_n(\leq n - 2\lfloor n/2 \rfloor)$$

with quotients

$$P_n(\leq n-2k)/P_n(\leq n-2(k+1)) \cong M_{n-2k}^{\widetilde{X}_{n-2k},n},$$

where $\widetilde{X}_{n-2k}, n$ denotes the collection of diagrams in ${}_nB_{n+2k}^C$ without left returns. Note that the cardinality of $\widetilde{X}_{n-2k}, n$ is $\frac{n+1}{n+k+1}\binom{n+2k}{k}$. In the Grothendieck group $G_0(A^C\text{-lfd})$ this filtration yields a relation:

$$[P_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n+1}{n+k+1}\binom{n+2k}{k}[M_{n-2k}].$$

One can also construct a projective resolution of M_n by P_k's:

$$0 \longrightarrow \ldots \longrightarrow P_{n-2k}^{\binom{n-k}{k}} \longrightarrow \ldots \longrightarrow P_{n-2}^{n-1} \longrightarrow P_n \longrightarrow M_n \longrightarrow 0$$

yielding the relation

$$[M_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}[P_{n-2k}] \tag{5}$$

in the Grothendieck group. Under the identification of $[P_n]$ with x^n, the relation (5) shows that $[M_n]$ can be identified with the Chebyshev polynomial (compare with (4)).

3.2 Hermite Polynomials

The Hermite polynomials are defined on the interval $(-\infty, \infty)$ and are orthogonal with respect to the inner product $(f,g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}}dx$. For this inner product we have, $(x^n, x^m) = (n+m-1)!!$ when $n+m$ is even and is equal to 0 otherwise. Thus (x^n, x^m) is counted by the number of perfect pairings of an $(n+m)$-element set.

Definition 3.2. *An (n,m)-Hermite diagram is an isotopy class of planar diagrams D in the strip $[0,1]\times\mathbb{R} \subseteq \mathbb{R}^2$ with:*

1. *n vertices on the line $x = 0$*
2. *m vertices on the line $x = 1$*
3. *Each vertex is the endpoint of an arc in D*
4. *Arcs in D are only allowed to begin and end at one of the $n+m$ vertices in $\{0,1\}\times\mathbb{R}$*
5. *Arcs beginning at $x = 0$ and ending at $x = 1$ (call these through arcs) must have well defined slope everywhere*

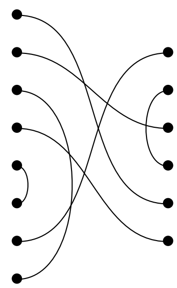


Figure 3: An example of an (n, m)-Hermite diagram

6. *Arcs beginning and ending at the same x value (call these returns) have exactly one point with undefined slope*
7. *Arcs have no self intersections*
8. *Each pair of arcs has at most one intersection*
9. *Only isotopies preserving the above conditions are allowed*

Let ${}_nB^H_m$ denote the set of all (n, m)-Hermite diagrams, and $B^H_m = \coprod_{n\in\mathbb{N}} {}_nB^H_m$. Let $\Bbbk$ be a field, let ${}_nA^H_m$ be the $\Bbbk$-vector space with basis ${}_nB^H_m$, and define

$$A^H = \bigoplus_{n,m\geq 0} {}_nA^H_m.$$

Endow A^H with a non-unital algebra structure by defining multiplication as follows:

- Given $x \in {}_nB^H_m$ and $y \in {}_mB^H_\ell$, define the product $xy \in {}_nB^H_\ell$ by concatenating horizontally. If the product xy defined in this way yields a diagram which does not satisfy all conditions in Definition 3.2, set $xy = 0$.
- Given $x \in {}_nB^H_m$ and $y \in {}_kB^H_\ell$ with $m \neq k$, set $xy = 0$.
- Extend to a multiplication on A^H by extending linearly from the action on the basis elements.

The algebra A^H has a family of mutually orthogonal idempotent elements $\{1_n\}_{n\in\mathbb{N}}$ where 1_n is the unique element of ${}_nB^H_n$ with no crossings. Define $P_n = A1_n$. The modules $\{P_n\}_{n\in\mathbb{N}}$ are projective and locally finite dimensional (meaning 1_mP_n is finite dimensional for any $m \geq 0$). Let A^H-plfd denote the category of projective locally finite dimensional A^H-modules. Any object in A^H-plfd is isomorphic to a finite direct sum of P_n's and the Grothendieck group $K_0(A^H\text{-plfd})$ is free abelian with basis $\{[P_n]\}_{n\in\mathbb{N}}$, and therefore $K_0(A^H\text{-plfd}) \cong \mathbb{Z}[x]$ via the map $[P_n] \mapsto x^n$. Let $\mathcal{C}(A^H\text{-plfd})$ denote the category of chain complexes over A^H-plfd. There is a canonical isomorphism $K_0(\mathcal{C}(A^H\text{-plfd})) \cong K_0(A^H\text{-plfd})$ by sending the symbol of a complex to its Euler characteristic $[X] \mapsto \sum_{i\in\mathbb{Z}}(-1)^i[X^i]$.

Let J be the ideal in A spanned by diagrams with at least one right return. For $n \in \mathbb{N}$ define the *big standard* module $\widetilde{M}_n$ as the quotient of P_n by the ideal in P_n spanned by diagrams with at least one right return. The big standard modules $\widetilde{M}_n$ have a basis of diagrams in B_n with no right returns. The standard modules M_n are defined as the quotient of $\widetilde{M}_n$ by the ideal spanned by diagrams with at least one intersection among through arcs, so M_n has a basis of diagrams in P_n with no right returns and no crossings among through arcs.

There is a filtration of $\widetilde{M}_n$ by submodules $\widetilde{M}_n^{(m)}$ spanned by diagrams d with $\ell(d) \geq m$ for $m \in \{0, 1, \ldots, \binom{n}{2}\}$ where $\ell(d)$ is the length of the permutation associated to the through arcs of d:

$$\widetilde{M}_n = \widetilde{M}_n^{(0)} \supseteq \widetilde{M}_n^{(1)} \supseteq \cdots \supseteq \widetilde{M}_n^{(\binom{n}{2})} = 0.$$

Each quotient $\widetilde{M}_n^{(i)}/\widetilde{M}_n^{(i+1)}$ is isomorphic to a direct sum of j copies of M_n where j is the number of permutations of length i. Thus in $K_0(A^H\text{-lfd})$ we have the relation $[\widetilde{M}_n] = n![M_n]$.

There is another useful filtration, this time of P_n. We write

$$P_n = P_n(n) \supseteq P_n(n-2) \supseteq P_n(n-4) \supseteq \dots$$

where $P_n(n-2k)$ is the submodule of P_n spanned by diagrams in B_n with at most $n-2k$ through arcs. One can show that the quotient $P_n(n-2k)/P_n(n-2k-2)$ is isomorphic to a direct sum of $u_{n,k} = \frac{n!}{2^k k!(n-2k)!}$ copies of $\widetilde{M}_{n-2k}$ so in the Grothendeick group $K_0(A^H\text{-lfd})$ we find the relation

$$[P_n] = \sum_{k=0}^{\frac{n}{2}} \frac{n!}{2^k k!(n-2k)!}[\widetilde{M}_{n-2k}]. \tag{6}$$

There is also a projective resolution of the big standard module $\widetilde{M}_n$:

$$P_0 \longrightarrow P_2 \longrightarrow \dots \longrightarrow P_{n-2k}^{u_{n,k}} \longrightarrow \dots \longrightarrow P_{n-2}^{u_{n,1}} \longrightarrow P_n \longrightarrow \widetilde{M}_n$$

which yields the relation

$$[\widetilde{M}_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k u_{n,k}[P_{n-2k}] \tag{7}$$

in $K_0(A^H\text{-lfd})$. Under the identification of $[P_n]$ with x^n, equation (7) tells us that $[\widetilde{M}_n] = H_n(x)$, the nth Hermite polynomial. Equation 6 expresses a formula for x^n in terms of the Hermite polynomial basis.

4 Laguerre Polynomials

The Laguerre polynomial $L_n(x)$ is defined by applying the Gram Schmidt process to the standard basis $\{1, x, x^2, ...\}$ of $\mathbb{Q}[x]$ with respect to the inner product

$$(f,g) = \int_0^\infty f(x)g(x)e^{-x}dx. \tag{8}$$

We find that

$$L_n(x) = \sum_{m\geq 0} (-1)^m \binom{n}{m} \frac{x^m}{m!}. \tag{9}$$

Since the coefficient $\binom{n}{m}\frac{1}{m!}$ of x^m is not an integer it makes sense to view this as an exponential generating function with $\binom{n}{m}$ as the coefficient of $\frac{x^m}{m!}$. Thus to categorify Laguerre polynomials we should first categorify the ring of exponential polynomials $\mathbb{Z}^e[x] := \mathbb{Z}[\frac{x^m}{m!} \mid m \in \mathbb{N}]$ with product

$$\frac{x^m}{m!} \cdot \frac{x^n}{n!} = \binom{n+m}{n} \frac{x^{n+m}}{(n+m)!}. \tag{10}$$

The inner product (8) acts on basis elements $\frac{x^m}{m!}, \frac{x^n}{n!}$ as follows:

$$\left(\frac{x^m}{m!}, \frac{x^n}{n!}\right) = \int_0^\infty \frac{x^m}{m!}\frac{x^n}{n!}e^{-x}dx = \binom{n+m}{n}. \tag{11}$$

5 Categorifying Laguerre Polynomials

AC: get rid of most of this because its now in an earlier section

Let A^- denote the algebra of slarcs defined in [1] and A^--pmod the category of projective finitely generated left modules over A^-. Indecomposable projectives look like $P_n = A^- 1_n$ for idempotents 1_n. $\mathrm{Hom}(P_n, P_m)$ is generated by slarc diagrams with n left and m right endpoints. Thus

$$\dim \mathrm{Hom}(P_n, P_m) = \binom{n+m}{n}. \tag{12}$$

In [1] it is shown that $K_0(A^-) \cong \mathbb{Z}[x]$ via the map $[P_n] \mapsto x^n$. As groups, $\mathbb{Z}[x] \cong \mathbb{Z}^e[x]$ via $x^n \mapsto \frac{x^n}{n!}$. Thus the map $f : K_0(A^-) \to \mathbb{Z}^e[x]$ sending $[P_n] \mapsto \frac{x^n}{n!}$ is an isomorphism. Notice also that $\dim \mathrm{Hom}(P_n, P_m) = (\frac{x^m}{m!}, \frac{x^n}{n!})$ so A^--pmod categorifies the groups structure on $\mathbb{Z}^e[x]$ with the bilinear form (11).

The standard module M_n is the quotient of P_n by the submodule generated by all diagrams with a positive number of right sarcs. M_n has a basis consisting of diagrams with no right sarcs. Recall that in $K_0(A^-)$ we have the relations

$$[P_n] = \sum_{m \geq 0} \binom{n}{m} [M_m] \tag{13}$$

and

$$[M_n] = \sum_{m \geq 0} (-1)^{n+m} \binom{n}{m} [P_m]. \tag{14}$$

Define the *signed standard module*

$$N_n := M_n[n] \tag{15}$$

where $[n]$ denotes homological degree shift by n. In $K_0(A^-)$ we have

$$[N_n] = (-1)^n [M_n]. \tag{16}$$

Identify $K_0(A^-)$ with $\mathbb{Z}^e[x]$ via $[P_n] = \frac{x^n}{n!}$. Then in $K_0(A^-)$,

$$\begin{aligned} [N_n] &= \sum_{m \geq 0} (-1)^{m+2n} \binom{n}{m} [P_m] && (17) \\ &= \sum_{m \geq 0} (-1)^m \binom{n}{m} \frac{x^m}{m!} && (18) \\ &= L_n(x) && (19) \end{aligned}$$

so signed standard modules categorify Laguerre polynomials. We also have

$$[P_n] = \sum_{m \geq 0} \binom{n}{m} [M_m] = \sum_{m \geq 0} (-1)^m \binom{n}{m} [N_m] \tag{20}$$

which translates to

$$\frac{x^n}{n!} = \sum_{m \geq 0} (-1)^m \binom{n}{m} L_m(x), \tag{21}$$

the usual inverse relation for Laguerre polynomials.

6 Categorifying the Multiplicative Structure on $\mathbb{Z}^e[x]$

To categorify the ring structure on $\mathbb{Z}^e[x]$ we need a monoidal product on A^--pmod which agrees with the product (10) in $\mathbb{Z}^e[x]$. We set

$$P_n \boxtimes P_k = P_{n+k}^{\binom{[n+k]}{n}} \tag{22}$$

where $\binom{[n+k]}{n}$ is the set of subsets of $[n+k] = \{1, 2, ..., n+k\}$ of size n (a module raised to the power of a set is just the direct sum of copies of that module, one for each element of the set). On occasion we may use the shorthand notation

$$P_n \boxtimes P_k = P_{n,k}$$

Now, given $\alpha \in {}_nB_m$ and $\beta \in {}_kB_r$ we define $\alpha \otimes \beta$ to be the diagram for β stacked on top of the diagram for α, as defined in [1]. Then we wish to define

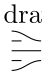
$$\alpha \boxtimes \beta : P_{n,k} \to P_{m,r}. \tag{23}$$

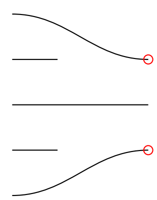
Instead of thinking of elements of $P_{n,k}$ as tuples indexed by $\binom{[n+k]}{n}$, we can think of them as formal linear combinations of the form

$$\sum_{S \in \binom{[n+k]}{n}} \gamma_S \cdot e^S \tag{24}$$

where $\gamma_S \in P_{n+k}$ (so the coefficient of e^S is the entry of $P_{n,k}$ corresponding to the subset S). We can use the Einstein summation convention to simply write

$$\gamma_S \cdot e^S = \sum_{S \in \binom{[n+k]}{n}} \gamma_S \cdot e^S \tag{25}$$

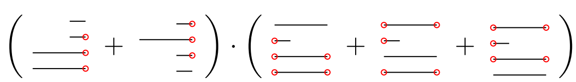
Diagrammatically, we can depict $\gamma \cdot e^S$ by drawing γ and circling the right endpoints which correspond to the elements of S. For example, given $\gamma =$  $\in P_3$ and $S = \{1, 3\}$ we could denote $\gamma \cdot e^S$ by the diagram



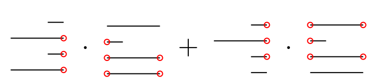
Similarly, a morphism in $\mathrm{Hom}\big(P_{n,k}, P_{m,r}\big)$ can be written as a formal sum

$$\gamma_{(R,P)} \cdot e^{(R,P)} = \sum_{(R,P) \in \binom{[n+k]}{n} \times \binom{[m+r]}{m}} \gamma_{(R,P)} \cdot e^{(R,P)} \tag{26}$$

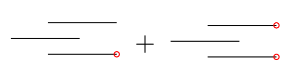
again using the summation convention when convenient. Here, the coefficient of $e^{(R,P)}$ maps the component of $P_{n,k}$ corresponding to the subset R into the component of $P_{m,r}$ corresponding to the subset P. Diagrammatically we denote $\alpha \cdot e^{(R,P)}$ by drawing the diagram α and circling left endpoints corresponding to R and right endpoints corresponding to P. Concatenation of circled diagrams is zero unless the gluing happens along the same subset of endpoints. Upon concatenation, we remove the circles on endpoints which have been glued together. For example,



yields



and upon concatenation we get



Notice that basis elements of triple products $(P_n \boxtimes P_k) \boxtimes P_\ell = P_{n+k+\ell}^{\binom{[n+k]}{n} \times \binom{[n+k+\ell]}{n+k}}$ are also indexed by pairs $(R, S) \in \binom{[n+k]}{n} \times \binom{[n+k+\ell]}{n+k}$ of subsets. So we may denote basis elements of $(P_n \boxtimes P_k) \boxtimes P_\ell$ by $\alpha \cdot e^{(R,S)}$ where $\alpha \in P_{n+k+\ell}$. However these pairs (R, S) consist of subsets which are both on the same side of the diagram. To alleviate some confusion, we let alphabetically adjacent pairs (P, Q), (R, S), (A, B) and (C, D) denote pairs of subsets on the same side of a diagram, and we let alphabetically nonadjacent pairs (R, P), (S, Q), (A, C) and (B, D) denote pairs on opposite sides of a diagram. We will make use of the shorthand notation

$$(P_n \boxtimes P_k) \boxtimes P_\ell = P_{(n,k),\ell}$$

Associating the other way we get $P_n \boxtimes (P_k \boxtimes P_\ell) = P_{n+k+\ell}^{\binom{[k+\ell]}{k} \times \binom{[n+k+\ell]}{n}}$. We will make use of the shorthand notation

$$P_n \boxtimes (P_k \boxtimes P_\ell) = P_{n,(k,\ell)}$$

The modules $P_{(n,k),\ell}$ and $P_{n,(k,\ell)}$ are isomorphic, but unfortunately the isomorphism is not natural.

To define the product $\alpha \boxtimes \beta$, there is an obvious diagram available, $\alpha \otimes \beta$ (that is, β stacked on top of α), which maps P_{n+k} into $P_{m+\ell}$. Thus it may make sense to define

$$\alpha \boxtimes \beta = f(S,T)(\alpha \otimes \beta) \cdot e^{(S,T)} \tag{27}$$

(summation implied) for some function $f : \binom{[n+k]}{n} \times \binom{[m+\ell]}{m} \to \mathbb{Z}$. One obvious choice would be $f \equiv 1$. This however would be somewhat trivial since then $\alpha \boxtimes \beta$ would have rank 1. Thus we make the following more interesting choice:

Notation 6.1. *A larc in a diagram $\alpha \in {}_nB_m$ will be denoted by the pair $(x,y) \in [n] \times [m]$ corresponding to its endpoints. We will write $(x,y) \in \alpha$ to denote that (x,y) is a larc in the diagram α, and $(R,S) \in \alpha$ to denote that (R,S) is a slarc subdiagram of α.*

Definition 6.2. *Given a slarc diagram $\alpha \in {}_nB_m$, a slarc subdiagram of α is a pair $(R,S) \in 2^{[n]} \times 2^{[m]}$ such that for each larc $(x,y) \in \alpha$ we have $x \in R \iff y \in S$.*

Definition 6.3. *Define the pairing $\left\langle \substack{\cdot \\ \cdot} \right\rangle : {}_nA_m \times (2^{[n]} \times 2^{[m]}) \to {}_nA_m$ to be linear in the top coordinate and such that for any $\alpha \in {}_nB_m$,*

$$\left\langle \begin{matrix} \alpha \\ (R,S) \end{matrix} \right\rangle = \begin{cases} \alpha & (R,S) \in \alpha \\ 0 & (R,S) \notin \alpha \end{cases}$$

Now, given $\alpha \in {}_nB_m$ and $\beta \in {}_kB_r$, define

$$\alpha \boxtimes \beta = \left\langle \begin{matrix} \alpha \otimes \beta \\ (R,S) \end{matrix} \right\rangle e^{(R,S)}$$

where there is an implied summation over all $(R,S) \in \binom{[n+k]}{n} \times \binom{[m+r]}{m}$.

This seems to be a natural and interesting choice since this actually depends on the diagram $\alpha \otimes \beta$ and the subsets S and T of the endpoints. Unfortunately, this choice does not give an associative product up to natural isomorphism. There may however still be a chance for this product to work. Our goal is to define a tensor product of chain complexes of projective A-modules which is associative. But the only chain complexes we are really interested in multiplying are the projective resolutions $P(M_n)$. In these projective resolutions, the only maps which appear are sums of elements of the form $e^{(S,S\setminus\{x\})}b_n^i$ where b_n^i is a slarc diagram consisting of n larcs and one sarc after the ith larc.

Lemma 6.4. *For all $n,m \geq 0$,*

$$1_n \boxtimes 1_m = 1_{P_{n+m}^{\binom{n+m}{n}}}.$$

Proof. By definition,

$$\begin{aligned} 1_n \boxtimes 1_m &= \left\langle \begin{matrix} 1_n \otimes 1_m \\ (R,S) \end{matrix} \right\rangle e^{(R,S)} \\ &= \left\langle \begin{matrix} 1_n \otimes 1_m \\ (R,R) \end{matrix} \right\rangle e^{(R,R)} \end{aligned}$$

since $\left\langle \substack{1_n \otimes 1_m \\ (R,S)} \right\rangle = 0$ for $R \neq S$. $\square$

Lemma 6.5. *There exist automorphisms ϕ, ψ for which*

$$\psi \circ \big((1_n \boxtimes 1_k) \boxtimes b\big) = \big(1_n \boxtimes (1_k \boxtimes b)\big) \circ \phi$$

$$\psi \circ \big((1_n \boxtimes b) \boxtimes 1_\ell\big) = \big(1_n \boxtimes (b \boxtimes 1_\ell)\big) \circ \phi$$

$$\psi \circ \big((b \boxtimes 1_k) \boxtimes 1_\ell\big) = \big(b \boxtimes (1_k \boxtimes 1_\ell)\big) \circ \phi$$

where $b = {}^ib_r$ or b_r^i. For b_n^i, ϕ is natural and ψ is not. For ${}^ib_n$, ψ is natural and ϕ is not.

Proof. In the three equations above, we always want n, k, ℓ to denote the number of larcs in the first, second, and third diagram respectively. Thus in the first equation, we let b have ℓ larcs, in the second we let b have k larcs and in the third we let b have n larcs. We start with the first equation. Note that

$$(1_n \boxtimes 1_k) \boxtimes b : P_{(n,k),\ell} \to P_{(n,k),\ell+1}$$

and

$$1_n \boxtimes (1_k \boxtimes b) : P_{n,(k,\ell)} \to P_{n,(k,\ell+1)}$$

Given objects a, b, c we think of $(ab)c$ to be the type 1 way to associate and $a(bc)$ to be the type 2 way to associate. For brevity we define

$$\begin{aligned} L_1 &= \binom{[n+k]}{n} \times \binom{[n+k+\ell]}{n+k} \\ R_1 &= \binom{[n+k]}{n} \times \binom{[n+k+(\ell+1)]}{n+k} \\ L_2 &= \binom{[k+\ell]}{k} \times \binom{[n+k+\ell]}{n} \\ R_2 &= \binom{[k+\ell+1]}{k} \times \binom{[n+k+(\ell+1)]}{n} \end{aligned}$$

To remember what these are, L_1 corresponds to the subsets of *left* endpoints indexing the type 1 way to associate the product. R_2 corresponds to the subsets of *right* endpoints indexing the type 2 way to associate the product (similarly for the rest). Notice then, that

$$P_{(n,k),\ell} = P^{L_1}_{n+k+\ell}$$

$$P_{(n,k),\ell+1} = P^{R_1}_{n+k+\ell+1}$$
$$P_{n,(k,\ell)} = P^{L_2}_{n+k+\ell}$$
$$P_{n,(k,\ell+1)} = P^{R_2}_{n+k+\ell+1}$$

We start by constructing the maps ϕ and ψ.

$$\begin{tikzcd} P_{(n,k),\ell} \arrow[r, "(1_n \boxtimes 1_k) \boxtimes b"] \arrow[d, "\phi"] & P_{(n,k),\ell+1} \arrow[d, "\psi"] \\ P_{n,(k,\ell)} \arrow[r, "1_n \boxtimes (1_k \boxtimes b)"] & P_{n,(k,\ell+1)} \end{tikzcd} \tag{28}$$

Note that we will always identify $[n]$ with the first n elements of $[n+k+l]$, $[k]$ with the elements $\{n+1, n+2, ..., n+k\}$, $[k+l]$ with the elements $\{n+1, n+1, ..., l\}$ and so on, so for example an element of $\binom{[n+k+l]}{n} \times \binom{[k+l]}{k}$ would consist of a choice of n elements from $[n+k+l]$ and a choice of k elements from $\{n+1, n+2, ..., l\}$. Notice that of course $|L_1| = |L_2|$ and so there is a bijection $\phi : L_1 \to L_2$.

We now construct an bijection

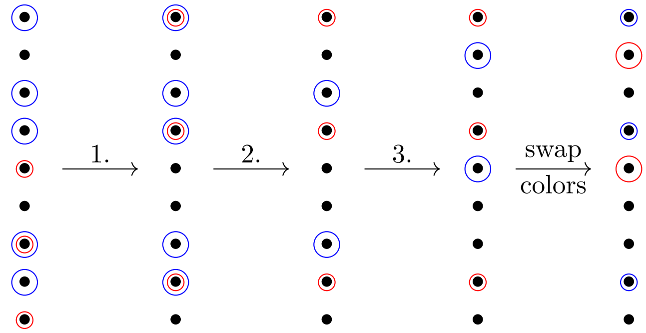
$$\phi = \phi_{n,k,\ell} : \binom{[n+k]}{n} \times \binom{[n+k+l]}{n+k} \to \binom{[k+l]}{k} \times \binom{[n+k+l]}{n}$$

which takes a pair (R, S) and does the following:

1. Consider the order preserving bijection of $[n+k]$ with T. Let R' be the image of R under this bijection, so $R' \subseteq S$.

2. Form the pair $(R', S \setminus R')$

3. Consider the order preserving bijection of $[n+k+l] \setminus S'$ with $[k+l]$. Let S' be the image of $S \setminus R'$ under this bijection and set $\phi(R, S) = (S', R')$.



ϕ induces an isomorphism

$$\phi : P^{L_1}_{n+k+\ell} \to P^{L_2}_{n+k+\ell}$$

(which we denote with the same symbol) by permuting components. That is

$$\phi(\alpha \cdot e^{(R,S)}) = \alpha \cdot e^{\phi(R,S)}$$

The definition of ψ will use the bijection ϕ and will also depend on the diagram b. Note that b has exactly one right sarc located at position x.

Define

$$\mathrm{Sub}_1 = \{(R, S, P, Q) \in L_1 \times R_1 \mid R \cup P, S \cup Q \in 1_n \otimes 1_k \otimes b_i^\ell\}$$

and

$$\mathrm{Sub}_2 = \{(A, B, C, D) \in L_2 \times R_2 \mid A \cup C, B \cup D \in 1_n \otimes 1_k \otimes b_i^\ell\}.$$

That is, Sub_1 and Sub_2 are the subcollections of $L_1 \times R_1$ and $L_2 \times R_2$ respectively consisting of tuples forming slarc subdiagrams.

Since $1_n \otimes 1_k \otimes b$ has only one sarc, (R, S) determines (P, Q) in Sub_1 and (A, B) determines (C, D) in Sub_2, and so the bijection $\phi : L_1 \to L_2$ determines a bijection $\phi' : \mathrm{Sub}_1 \to \mathrm{Sub}_2$. Now, define

$$R_1^x = \{(P, Q) \in R_1 \mid x \notin P \cup Q\}$$

$$R_2^x = \{(C, D) \in R_2 \mid x \notin C \cup D\}$$

Since any $(P, Q) \in R_1^x$ determines a slarc subdiagram, and similarly any $(C, D) \in R_2^x$ determines a slarc subdiagram, we have bijections $i_1^x : R_1^x \to \mathrm{Sub}_1$ and $i_2^x : R_2^x \to \mathrm{Sub}_2$. We get the following commutative diagram, in which any arrow labeled $\cong$ is a bijection:

$$\begin{tikzcd}
R_1 \arrow[r, dotted, "\exists\psi", "\cong"'] & R_2 \\
R_1^x \arrow[u, "i_1"] \arrow[r, dotted, "(i_2^x)^{-1}\circ\phi'\circ i_1^x", "\cong"'] \arrow[d, "i_1^x", "\cong"'] & R_2^x \arrow[u, "i_2"'] \arrow[d, "i_2^x", "\cong"'] \\
\mathrm{Sub}_1 \arrow[r, "\phi'", "\cong"'] \arrow[d, "p_1", "\cong"'] & \mathrm{Sub}_2 \arrow[d, "p_2", "\cong"'] \\
L_1 \arrow[r, "\phi", "\cong"'] & L_2
\end{tikzcd}$$

Since we know that $|R_1| = |R_2|$ and we have a bijection between $R_1^x \subseteq R_1$ and $R_2^x \subseteq R_2$ we can extend this to a bijection $\psi : R_1 \to R_2$ for which $x \in (P,Q) \iff x \in \psi(P,Q)$ where $x \in (P,Q)$ means $x \in P$ or $x \in Q$. Again we let ψ induce an isomorphism $\psi : P^{R_1}_{n+k+\ell+1} \to P^{R_2}_{n+k+\ell+1}$ by permuting coordinates according to ψ. That is, $\psi(\alpha \cdot e^{(P,Q)}) = \alpha \cdot e^{\psi(P,Q)}$

Now, to show that (28) commutes, consider the action on a diagram $\alpha \cdot e^{(R,S)} \in P^{L_1}_{n+k+\ell}$:

$$\begin{aligned}\big((1_n \boxtimes 1_k) \boxtimes b^i_\ell\big)\big(\alpha \cdot e^{(R,S)}\big) &= \sum_{(R,S,P,Q)\in \mathrm{Sub}_1} \alpha \cdot 1_n \otimes 1_k \otimes b^i_\ell \cdot e^{(P,Q)} \\ &= \alpha \cdot 1_n \otimes 1_k \otimes b^i_\ell \cdot e^{(P_0,Q_0)}\end{aligned}$$

where $(P_0, Q_0) = (i_1^x)^{-1} \circ p_1^{-1}(R,S)$ is the only pair for which (R,S,P_0,Q_0) is a slarc subdiagram.

Then applying the bijection ψ yields

$$\psi\Big(\big((1_n \boxtimes 1_k) \boxtimes b^i_\ell\big)\big(\alpha \cdot e^{(R,S)}\big)\Big) = \alpha \cdot 1_n \otimes 1_k \otimes b^i_\ell \cdot e^{\psi(P_0,Q_0)}$$

Going the other way, we begin by applying the bijection ϕ:

$$\phi\big(\alpha \cdot e^{(R,S)}\big) = \alpha \cdot e^{\phi(R,S)}$$

and then apply the 2nd way to associate the product:

$$\begin{aligned}\big(1_n \boxtimes (1_k \boxtimes b^i_\ell)\big)\big(\phi(\alpha \cdot e^{(R,S)})\big) &= \big(1_n \boxtimes (1_k \boxtimes b^i_\ell)\big)\big(\alpha \cdot e^{\phi(R,S)}\big) \\ &= \sum_{(\phi(R,S),C,D)\in \mathrm{Sub}_2} \alpha \cdot 1_n \otimes 1_k \otimes b^i_\ell \cdot e^{(C,D)} \\ &= \alpha \cdot 1_n \otimes 1_k \otimes b^i_\ell \cdot e^{(C_0,D_0)}\end{aligned}$$

where $(C_0, D_0) = (i_2^x)^{-1} \circ p_2^{-1} \circ \phi(R,S)$ is the only pair for which $(\phi(R,S), C_0, D_0)$ is a slarc subdiagram. Then $(1_n \boxtimes 1_k) \boxtimes b^i_\ell = 1_n \boxtimes (1_k \boxtimes b^i_\ell)$ since our commutative diagram of bijections tells us

$$\psi \circ (i_1^x)^{-1} \circ p_1^{-1} = (i_2^x)^{-1} \circ p_2^{-1} \circ \phi.$$

Nothing in this argument depended on b being in the third position, so a similar argument can be used to prove the other two equalities. For ${}^ib_n$ we would have to first define ψ explicitly and then implicitly define ϕ analogously to what we did above. □

Let $\mathcal{A}$ be the subcategory of A^--pmod generated by objects P_n, morphisms 1_n, b^i_n, ${}^ib_n$ for all $i \le n \in \mathbb{N}$ and all possible direct sums and tensor products of these. AC: Maybe not the right approach... What we do have is a product on A^--pmod which is associative on $\mathcal{C}$ so we can define tensor products of the resolutions $P(M_n)$. However this product on the category of chain complexes may not yield exactly the Grothendieck ring $\mathbb{Z}^e[x]$.

Questions: Is $\mathcal{A}$ abelian? What is $K_0(\mathcal{A})$? What is $K_0(\mathcal{C}(\mathcal{A}))$? Are they equal? How does $K_0(\mathcal{C}(\mathcal{A}))$ relate to $K_0(\mathcal{C}(A^-\text{-pmod}))$?

Lemma 6.6. *For any morphisms $c, d, e \in \mathcal{A}$, there are automorphisms ϕ, ψ for which $\psi \circ [c \boxtimes (d \boxtimes e)] = [(c \boxtimes d) \boxtimes e] \circ \phi$.* AC: *This is obviously false and not necessary... keeping here for now to recall general idea*

Proof. Represent c, d, e in matrix form: $c = (c_{i,j})$, $d = (d_{k,\ell})$, $e = (e_{r,s})$ where each matrix element is $1_n, b^i_n$, or ${}^ib_n$ (could even be sums of these in general). We have

$$[(c \boxtimes d) \boxtimes e]_{a,b} = \sum_k \sum_\ell (c_{a,k} \boxtimes d_{k,\ell}) \boxtimes e_{\ell,b}$$

and

$$[c \boxtimes (d \boxtimes e)]_{a,b} = \sum_k \sum_\ell c_{a,k} \boxtimes (d_{k,\ell} \boxtimes e_{\ell,b})$$

□

Conjecture 6.7. *The bifunctor* $\boxtimes$ *on* A^--pmod *extends to a bifunctor on* $\mathcal{C}(A^-\text{-pmod})$ *(in the usual way) which is associative up to isomorphism on the subcategory* $\mathcal{C}(\mathcal{A})$*, and* $\mathcal{C}(\mathcal{A})$ *contains all of the resolutions* $P(M_n)$ *for* $n \geq 0$*.*

Proof. Given complexes $A, B, C \in \mathcal{C}(\mathcal{A})$ we will exhibit an isomorphism $A \boxtimes (B \boxtimes C) \cong (A \boxtimes B) \boxtimes C$. We have

$$[A \boxtimes (B \boxtimes C)]^\ell = \bigoplus_{i+j+k=\ell} A^i \boxtimes (B^j \boxtimes C^k)$$

$$[(A \boxtimes B) \boxtimes C]^\ell = \bigoplus_{i+j+k=\ell} (A^i \boxtimes B^j) \boxtimes C^k.$$

Let d_1, d_2 denote the differentials on $(A \boxtimes B) \boxtimes C$ and $A \boxtimes (B \boxtimes C)$ respectively. Lemma 6.5 gives automorphisms ϕ_i, ψ_i such that $\psi_i \circ d_1 = d_2 \circ \phi_i$. (does it?)

□

7 Weak Monoidal Structure And Why It's Enough

AC: "weak monoidal" is already used terminology, need to change this. Also, the product structure here doesn't even give a weak monoidal category. It only associates on a certain subcategory $\mathcal{C}$. So this section is probably not needed, or needs to be modified to be more general.

Given an additive category $\mathcal{C}$ with binary products of objects M, N denoted $M \oplus N$ we can define the Grothendieck group $K_0(\mathcal{C})$ as the abelian group generated by symbols $[M]$ over all objects $M \in \mathrm{Ob}(\mathcal{C})$ and relations $[M] = [N] + [K]$ whenever $M \cong N \oplus K$ in $\mathcal{C}$.

Definition 7.1. *A* ***monoidal category*** *is a category* $\mathcal{C}$ *together with a bifunctor* $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *which has*

1. *An object* $1 \in Ob(\mathcal{C})$ *and natural isomorphisms* $1 \boxtimes M \cong M \cong M \boxtimes 1$
2. *Natural isomorphisms* $(M \boxtimes N) \boxtimes K \cong M \boxtimes (N \boxtimes K)$
3. *Coherence conditions (triangle diagram and pentagon diagram)*

If $(\mathcal{C}, \boxtimes)$ is an additive monoidal category, $K_0(\mathcal{C})$ can be endowed with the structure of a ring by setting $[M]\cdot[N] = [M \boxtimes N]$. In this note we consider which of the above properties are necessary for the construction of Grothendieck ring. We will find that the following is sufficient:

Definition 7.2. *A* ***weak monoidal category*** $\mathcal{C}$ *is a category with a bifunctor* $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *which has*

1.′ *An object* $1 \in Ob(\mathcal{C})$ *such that* $1 \boxtimes M \cong M \cong M \boxtimes 1$

2.′ *Existence of isomorphisms* $(M \boxtimes N) \boxtimes K \cong M \boxtimes (N \boxtimes K)$

Proposition 7.3. *Given an additive weak monoidal category* $(\mathcal{C}, \boxtimes)$*, for which* $\boxtimes$ *is an additive bifunctor, defining*

$$[M] \cdot [N] = [M \boxtimes N]$$

endows $K_0(\mathcal{C})$ *with a unital associative ring structure.*

Proof. We already know that $K_0(\mathcal{C})$ has the structure of an abelian group via $[M] + [N] = [M \oplus N]$. We must show that the multiplicative structure is unital, associative, and distributive.

1. *Unital*: Since $M \boxtimes 1 \cong M \cong 1 \boxtimes M$ in $\mathcal{C}$ we get the relations $[1] \cdot [M] = [M] = [M] \cdot [1]$ in $K_0(\mathcal{C})$.
2. *Associative*: Since $(M \boxtimes N) \boxtimes K \cong M \boxtimes (N \boxtimes K)$ in $\mathcal{C}$ we get

$$\begin{aligned}
([M] \cdot [N]) \cdot [K] &= [M \boxtimes N] \cdot [K] \\
&= [(M \boxtimes N) \boxtimes K] \\
&= [M \boxtimes (N \boxtimes K)] \\
&= [M] \cdot [N \boxtimes K] \\
&= [M] \cdot ([N] \cdot [K])
\end{aligned}$$

3. *Distributivity*: Since $\boxtimes$ is additive, we have

$$M \boxtimes (N \oplus K) \cong (M \boxtimes N) \oplus (M \boxtimes K)$$

and therefore

$$\begin{aligned}[M] \cdot ([N] + [K]) &= [M] \cdot [N \oplus K] \\ &= [M \boxtimes (N \oplus K)] \\ &= [(M \boxtimes N) \oplus (M \boxtimes K)] \\ &= [M \boxtimes N] + [M \boxtimes K] \\ &= ([M] \cdot [N]) + ([M] \cdot [K]).\end{aligned}$$

Similarly, one can show that right multiplication distributes.

□

Proposition 7.4. *(A^--pmod, $\boxtimes$) is an additive weak monoidal category and $\boxtimes$ is an additive bifunctor.*

Proof. First we notice that P_0 acts as a monoidal unit:

$$P_0 \boxtimes P_n = P_n^{\binom{[n]}{0}} = P_n$$

and similarly

$$P_n \boxtimes P_0 = P_n^{\binom{[n]}{n}} = P_n.$$

We have already seen an isomorphism

$$(P_n \boxtimes P_m) \boxtimes P_k \cong P_n \boxtimes (P_m \boxtimes P_k).$$

$\boxtimes$ is an additive bifunctor since it was defined by additive extension.

□

When an abelian category $\mathcal{A}$ has a monoidal structure (in which $(A, B) \mapsto A \boxtimes B$ is additive in each variable), a monoidal structure is induced on the homotopy category $K^b(A)$ of bounded chain complexes via

$$P \boxtimes Q = \bigoplus_{i+j=n} P^i \boxtimes Q^j$$

with horizontal differentials $d \boxtimes 1$ and vertical differentials $(-1)^i \boxtimes d$. This also works on the weak level, however with some additional assumptions.

Proposition 7.5. *Suppose that ($\mathcal{A}$, $\boxtimes$) is a weak monoidal abelian category, and there are isomorphisms $\phi_1, \ldots, \phi_6$ such that the following holds:*

$$\phi_1 \circ 1_x \boxtimes (f \boxtimes 1_y) = (1_x \boxtimes f) \boxtimes 1_y \circ \phi_2$$
$$\phi_3 \circ (1_x \boxtimes 1_y) \boxtimes f = 1_x \boxtimes (1_y \boxtimes f) \circ \phi_4$$
$$\phi_5 \circ f \boxtimes (1_x \boxtimes 1_y) = (f \boxtimes 1_x) \boxtimes 1_y \circ \phi_6$$

for any objects x, y and any morphism f. Then ($K^b(\mathcal{A})$, $\boxtimes$) is also weak monoidal using the definition provided above. Furthermore, we have

$$K_0(K^b(\mathcal{A})) \cong K_0(\mathcal{A})$$

as rings via the isomorphism

$$[C] \mapsto \sum_{n \in \mathbb{Z}} (-1)^n C^n.$$

(probably need some extra conditions such as $\mathcal{A}$ is Noetherian and Artinian, and all objects in $\mathcal{A}$ are projective)

Proof. Weak monoidal structure on $K^b(\mathcal{A})$: Associativity holds on the level of chain groups since it holds for objects in ($\mathcal{A}$, $\boxtimes$). We must show that chain maps are the same for $(A^\bullet \boxtimes B^\bullet) \boxtimes C^\bullet$ and $A^\bullet \boxtimes (B^\bullet \boxtimes C^\bullet)$. This follows exactly from the assumed associativity equations. □

8 Categorifying Products of Laguerre Polynomials

In the identification $[P_m] = x^m$ and the corresponding monoidal structure which categorifies $x^n x^m = x^{n+m}$ standard modules M_m correspond to polynomials $(x-1)^m$. The product

$$(x-1)^m (x-1)^n = (x-1)^{n+m}$$

is then categorified by the isomorphism

$$P(M_m) \otimes P(M_n) \cong P(M_{n+m})$$

where $P(M_n)$ is the projective resolution $P(M_n) \to M_n$:

$$0 \to P_0 \to \cdots \to P_{n-m}^{\binom{n}{m}} \to \cdots \to P_{n-2}^{\binom{n}{2}} \to P_{n-1}^{\binom{n}{1}} \to P_n \to \underline{M_n}$$

described in proposition 2.9 in [1] where M_n is in homological degree 0. Similarly, the signed standard module N_n has a projective resolution:

$$0 \to \underline{P_0} \to \cdots \to P_{n-m}^{\binom{n}{m}} \to \cdots \to P_{n-2}^{\binom{n}{2}} \to P_{n-1}^{\binom{n}{1}} \to P_n \to M_n$$

where M_n is in homological degree n.

We would like a similar story for the identification $[P_m] = \frac{x^m}{m!}$ and its corresponding (weak) monoidal structure. In particular, in [2] it is shown that

$$L_r(x) L_s(x) = \sum_{t=|r-s|}^{r+s} C_{rst} L_t(x)$$

where C_{rst} is the integer

$$C_{rst} = (-1)^p \sum_n 2^{2n-p} \frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!}$$

and where $p = r+s-t$. Notice that the term $\frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!}$ is a multinomial coefficient:

$$\frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!} = \binom{r+s-n}{r-n, s-n, 2n-p, p-n}.$$

There are two ways (at least) that we can interpret the product of Laguerre polynomials in this categorified setting. First we can rewrite the product as

$$(-1)^r L_r(x) \cdot (-1)^s L_s(x) = \sum_{t=|r-s|}^{r+s} c_{rst} (-1)^t L_t(x)$$

where $c_{rst} = |C_{rst}|$. Then, since $[M_n] = (-1)^n L_n(x)$ we interpret this as

$$[M_r] \cdot [M_s] = \sum_{t=|r-s|}^{r+s} c_{rst} [M_t].$$

This might suggest that there is a filtration

$$P(M_r) \boxtimes P(M_s) = F_{r+s} \supset F_{r+s-1} \supset \cdots \supset F_{|r-s|}$$

where F_i/F_{i-1} is isomorphic to c_{rst} copies of our projective resolution of the standard module $P(M_t)$. Alternatively, we could write

$$L_r(x) \cdot L_s(x) = \sum_{t=|r-s|}^{r+s} c_{rst} (-1)^{r+s+t} L_t(x),$$

viewing $L_n(x)$ instead as coming from the signed standard module $[N_n] = L_n(x)$. Then we find

$$[N_r] \cdot [N_s] = \sum_{t=|r-s|}^{r+s} (-1)^{r+s+t} c_{rst} [N_t].$$

This may lead us to suspect that there is a resolution

$$0 \to P(N_{|r-s|})^{c_{rs|r-s|}} \cdots \to P(N_{r+s-1})^{c_{rs(r+s-1)}} \to P(N_{r+s})^{c_{rs(r+s)}} \to P(N_r) \boxtimes P(N_s)$$

9 Associative tensor product

Let us consider the more general tensor product between $\alpha \in {}_nB_m$ and $\beta \in {}_kB_\ell$ defined by $\alpha \boxtimes \beta : P_{n+k}^{\binom{[n+k]}{n}} \to P_{m+\ell}^{\binom{[m+\ell]}{m}}$

$$(\alpha \boxtimes \beta)(\gamma_S)_{S \in \binom{[n+k]}{n}} = (\sum_S f(S,T) \gamma_S \cdot \alpha \otimes \beta)_{T \in \binom{[m+\ell]}{m}} \tag{29}$$

where $f : \binom{[n+k]}{n} \times \binom{[m+\ell]}{m} \to \mathbb{Z}$. In order for $\boxtimes$ to be associative, we must have

$$f(S,T) f(R,M) = f(\phi(S,T)) f(\phi(R,M)) \tag{30}$$

for any $(S,R,T,M) \in \binom{[n+k+l]}{n+k} \times \binom{[n+k]}{n} \times \binom{[n+p+l]}{n+p} \times \binom{[n+p]}{n}$.

We now consider the case $f(S,T) = 1$ for all S,T which trivially satisfies the above condition. Diagrammatically, $\alpha \boxtimes \beta$ is the sum over all possible labelings of $\alpha \otimes \beta$ by subsets $S \subseteq [n+k]$, and $T \subseteq [m+\ell]$ where $|S| = n$ and $|T| = m$.

References

[1] M. Khovanov and R. Sazdanovic, "Categorifications of the polynomial ring," *Fundamenta Mathematicae*, vol. 230, no. 3, pp. 251–280, 2015.

[2] J. Gillis and G. Weiss, "Products of laguerre polynomials," *Mathematics of Computation*, vol. 14, no. 69, pp. 60–63, 1960.