Thin Posets and Homology Theories

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Outline

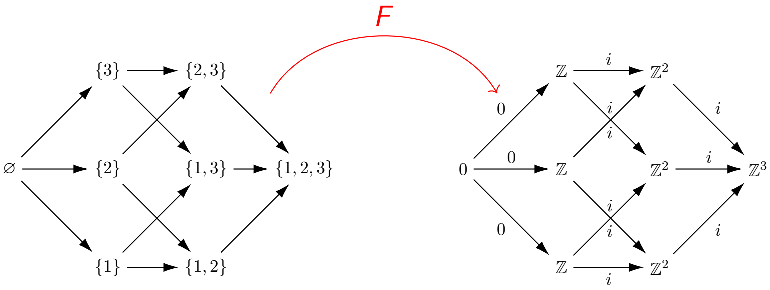
Categories and Functors

Category	Objects	Morphisms
FinSet	finite sets	functions
$\mathbb{k}$-**Vect**	$\mathbb{k}$-vector spaces	$\mathbb{k}$-linear maps
$\mathcal{C}^b(\mathbb{k}$-**Vect**$)$	bounded chain complexes of $\mathbb{k}$-v.s.'s	chain maps
$\mathbb{k}$-**gVect**	graded $\mathbb{k}$-v.s. $V = \oplus_{i\in\mathbb{Z}} V_i$.	graded linear maps
$\mathcal{C}^b(\mathbb{k}$-**gVect**$)$	chain complexes of graded $\mathbb{k}$-vector spaces	graded chain maps

A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ sends objects in $\mathcal{C}$ to objects in $\mathcal{D}$, morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$, and respects compositions and identity morphisms.

Posets as Categories

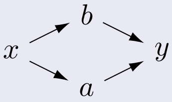
- Any poset $(P, \leq)$ can be thought of as a category: with objects P and a unique morphism from x to y iff $x \leq y$.
- A functor on a poset is a labeling of nodes and edges of the Hasse diagram by objects and morphisms so that compositions along any two paths between the same two nodes agree.



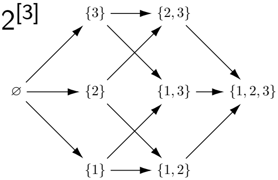
Thin Posets

Definition

A graded poset is *thin* if every nonempty interval $[x, y]$ with $\mathrm{rk}(y) - \mathrm{rk}(x) = 2$ is a diamond:

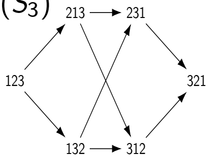


E.g. $2^{[3]}$

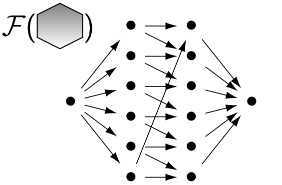


Boolean lattices

$\mathrm{Br}(S_3)$



Bruhat orders



Face posets of polytopes

Functors on Thin Posets Yield Cohomology Theories

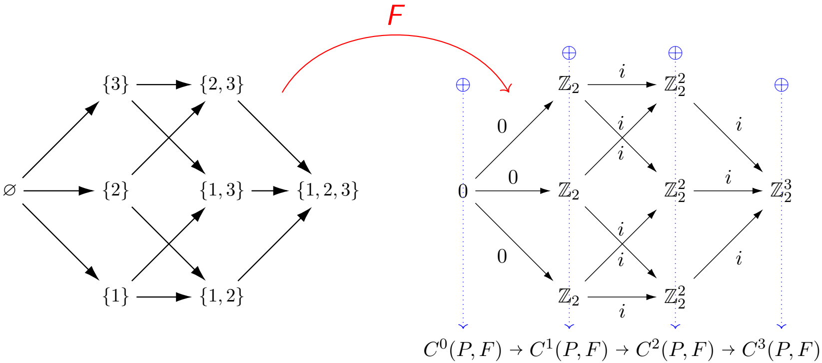
Given a thin poset P and a functor $F : P \to \mathbb{Z}_2$-**gVect**, define a cochain complex $C^*(P, F)$ by

$$C^k(P, F) = \bigoplus_{\mathrm{rk}(x)=k} F(x)$$

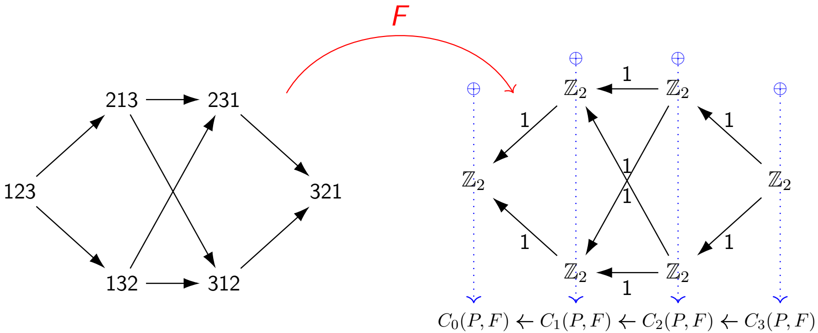
$$d^k : C^k(P, F) \to C^{k+1}(P, F) \qquad d^k = \sum_{\substack{x \lessdot y \\ \mathrm{rk}(x)=k}} F(x \lessdot y)$$

Since F commutes on diamonds, it follows that $d^2 = 0$. Denote the cohomology by $H^*(P, F)$.

Thin Poset Cohomology Pictorially



Contravariant Functors Yield Homology Theories

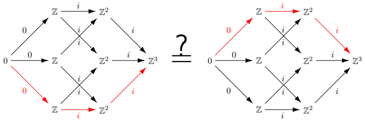


Constructing Functors on Thin Posets

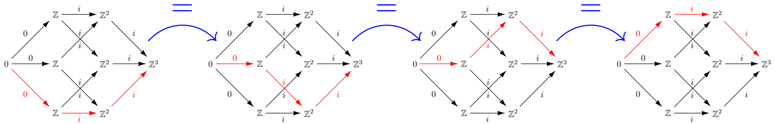
Question

If P is thin, does any labeling of the Hasse diagram with objects and morphisms which commutes on diamonds determine a functor?

E.g. Suppose we know diamonds commute.
Can we show these are equal?



Idea: commute one diamond at a time



Diamond Moves on Maximal Chains

Definition

A diamond

$$d = \begin{array}{ccccc} & \nearrow & b & \searrow & \\ x & & & & y \\ & \searrow & a & \nearrow & \end{array}$$

in P is *compatible* with a maximal chain C in P if C contains either $x \lessdot a \lessdot y$ or $x \lessdot b \lessdot y$. Define dC to be the maximal chain gotten by swapping a and b if d and C are compatible and otherwise setting $dC = C$. Call the action $C \mapsto dC$ a *diamond move* on C.

Notice that $d^2 C = C$ for any diamond d.

The Diamond Group

- P a thin poset, $S = \{\text{diamonds in } P\}$
- $w(S) = \langle S \mid d^2 = e, d \in S \rangle$
- $\mathcal{C}_{x,y}$ the set of maximal chains in $[x, y] \subseteq P$ and $\mathcal{C} = \amalg_{x \leq y} \mathcal{C}_{x,y}$
- $w(S)$ acts on $\mathcal{C}$ via diamond moves
- $N = \langle w \in w(S) \mid wC = C \ \forall \ C \in \mathcal{C} \rangle$ normal subgroup

Definition

The *diamond group* is $D(P) := w(S)/N$.

Note: $D(P)$ acts effectively on $\mathcal{C}$ via diamond moves.

Question

Which kinds of groups can arise as diamond groups of thin posets?

Diamond Transitivity

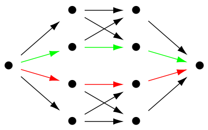
Definition

Say a thin poset P is *diamond transitive* if $D(P)$ acts transitively on $\mathcal{C}_{x,y}$ for each $x \leq y$ in P.

Question

Are all thin posets diamond transitive?

No! Counter example:



i.e. pinch two thin posets together of rk ≥ 3

Question

Is this the only type of obstruction?

Question: Which thin posets are diamond transitive?

Theorem (C., Hollering, Lacina 2018)

Face posets of simplicial complexes are diamond transitive.

Proof: Since intervals are boolean lattices, WLOG it suffices to consider $2^{[n]}$ and maximal chains between $\varnothing$ and $[n]$. Each rank is totally ordered lexicographically, and this induces a total order on maximal chains from $\varnothing$ to $[n]$. Let C_0 denote the lex largest chain $\varnothing \subseteq \{n\} \subseteq \{n-1, n\} \subseteq \cdots \subseteq [n]$. Suppose $C < C_0$ in lex order and induct on $d(C, C_0)$ to show C can be taken to C_0 by diamond moves. Since C is not lex largest, there is a subchain $S \subseteq S + i \subseteq S + i + j$ of C with $i < j$. Perform the diamond move replacing this subchain with $S \subseteq S + j \subseteq S + i + j$, which is lex larger so we are done by induction.

Question: Which thin posets are diamond transitive?

Definition (Björner, 1984)

A *CW poset* is any poset isomorphic to a face poset of a regular CW complex.

- Remark: CW posets are thin
- Examples of CW posets include
  - face posets of simplicial complexes (thus Boolean lattices)
  - face posets of polytopes
  - Bruhat orders on Coxeter groups
  - Any thin shellable poset

Question

Are all CW posets diamond transitive?

Extension to Abelian categories

Given a thin poset and a functor $F : P \to \mathcal{A}$ where $\mathcal{A}$ is an abelian category, $C^*(P, F)$ (as defined previously) is not a chain complex.

E.g. each diamond $A \xrightarrow{f} B \xrightarrow{g} D$, $A \to C \to D$ contributes $2gf$ to d^2. To fix this, we introduce signs to make diamonds anticommute. Let $C(P)$ denote the set of cover relations in P (i.e. edges in the Hasse diagram).

Definition

A *balanced coloring* of a thin poset P is a function $c : C(P) \to \{+1, -1\}$ such that each diamond has an odd number of -1's.

Thin Poset Homology over Abelian Categories

Given a thin poset P with balanced coloring c and a functor $F : P \to \mathcal{A}$, define

$$C^k(P,F)_c = \bigoplus_{\mathrm{rk}(x)=k} F(x)$$

$$d^k : C^k(P,F)_c \to C^{k+1}(P,F)_c \qquad d^k = \sum_{\substack{x \lessdot y \\ \mathrm{rk}(x)=k}} c(x \lessdot y) F(x \lessdot y)$$

Since cF anti-commutes on diamonds, it follows that $d^2 = 0$.

More Questions

Question

Are all thin posets balanced colorable?

Remark: the construction of cellular homology guarantees that all CW posets are balanced colorable. However this gives us only existence. Combinatorial formulas are still needed.

Question

Is the homology independent of the choice of balanced coloring?

Categorification (a philosophy)

Categorification is the concept of finding category theoretic analogues of set theoretic or algebraic structures:

categorification →

sets	categories
elements	objects
functions	functors
equations between elements	isomorphisms between objects

← decategorification

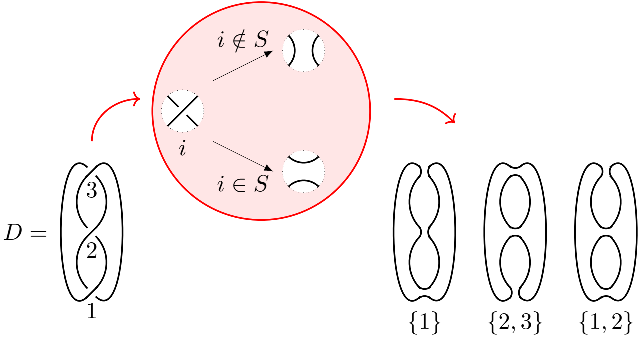
Decategorification is the reverse process (forgetting the extra structure)

Categorification Dictionary

Set, structure	Categorification	How to decategorify
$\mathbb{N}, +, \cdot$	**FinSet**, $\amalg, \times$ S	cardinality $\lvert S \rvert$
$\mathbb{N}, +, \cdot$	$\Bbbk$-**Vect**, $\oplus, \otimes$ V	dimension $\dim V$
$\mathbb{Z}, +, \cdot$	$\mathcal{C}^b(\Bbbk\text{-}\mathbf{Vect}), \oplus, \otimes$ $C = \bigoplus_{n \in \mathbb{Z}} C_n$	Euler characteristic $\chi(C) = \sum_{n \in \mathbb{Z}} (-1)^n \dim C_n$
$\mathbb{N}[q, q^{-1}], +, \cdot$	$\Bbbk$-**gVect**, $\oplus, \otimes$ $V = \bigoplus_{n \in \mathbb{Z}} V_n$	graded dimension $q\dim V = \sum_{n \in \mathbb{Z}} q^n \dim V_n$
$\mathbb{Z}[q, q^{-1}], +, \cdot$	$\mathcal{C}^b(\Bbbk\text{-}\mathbf{gVect}), \oplus, \otimes$ $C = \bigoplus_{n \in \mathbb{Z}} C_n$	graded Euler characteristic $\chi_q(C) = \sum_{n \in \mathbb{Z}} (-1)^n q\dim C_n$

A classic example from knot theory

- D a knot diagram with crossings $X = \{1, \ldots, n\}$
- Each $S \in 2^X$ encodes a *resolution* of D



The Jones polynomial for knots/links

The *Jones polynomial* (up to rescaling) has a state sum:

$$J(D) = \sum_{S \in 2^X} (-1)^{|S|} q^{|S|} (q + q^{-1})^{j(S)}$$

where $j(S)$ is the number of disjoint circles in the resolution corresponding to S.

Categorified Jones Polynomial (Khovanov Homology)

$$J(D) = \sum_{S \in 2^X} (-1)^{|S|} q^{|S|} (q + q^{-1})^{j(S)}$$

In 1999, Khovanov defined a functor $F_{\mathrm{Kh}} : 2^X \to \mathbb{Z}_2\text{-}\mathbf{gVect}$ with

- $q \dim F(S) = q^{|S|}(q + q^{-1})^{j(S)}$
- $\chi_q C^*(2^X, F_{\mathrm{Kh}}) = \chi_q H^*(2^X, F_{\mathrm{Kh}}) = J(D)$

Thus Khovanov categorified the Jones polynomial, obtaining a homological invariant, leading to numerous discoveries within knot theory, and many connections to other invariants in low dimensional topology and physics.

Another example: the Vandermonde determinant

Given $\vec{s} \in \mathbb{Z}_+^n$, the corresponding **generalized Vandermonde determinant** is:

$$V_{\vec{s}}(\vec{x}) = \begin{vmatrix} x_1^{s_1} & x_1^{s_2} & \cdots & x_1^{s_n} \\ x_2^{s_1} & x_2^{s_2} & \cdots & x_2^{s_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{s_1} & x_n^{s_2} & \cdots & x_n^{s_n} \end{vmatrix} = \sum_{\pi \in S_n} (-1)^{\text{inv}(\pi)} x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \ldots x_n^{s_{\pi(n)}}$$

- S_n has a thin partial order (Bruhat order)
- The Bruhat order is ranked by $\text{inv}(\pi)$

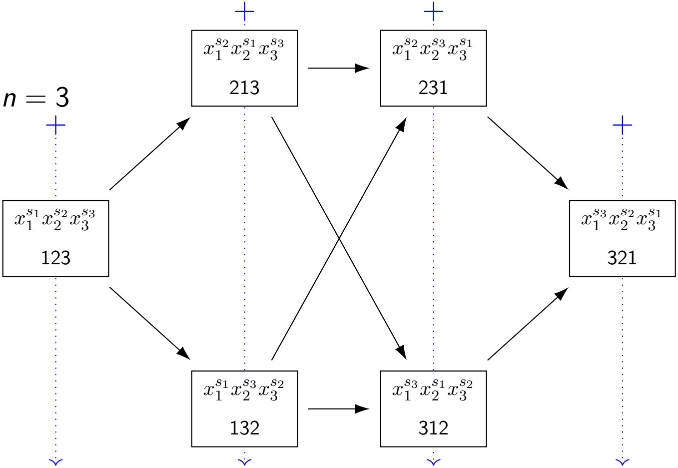
How to Categorify $V_{\vec{s}}(\vec{x})$

$$V_{\vec{s}}(\vec{x}) = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \dots x_n^{s_{\pi(n)}}$$

- Let $\vec{x} \in \mathbb{Z}_+^n$
- Define $\mathbb{Z}_2$-vector spaces $A_i = (\mathbb{Z}_2)^{x_i}$ so $\dim A_i = x_i$
- For $\pi \in S_n$, set $A_\pi = A_1^{\otimes s_{\pi(1)}} \otimes \cdots \otimes A_n^{\otimes s_{\pi(n)}}$
- $\dim A_\pi^{\vec{s}} = x_1^{s_{\pi(1)}} x_2^{s_{\pi(2)}} \dots x_n^{s_{\pi(n)}}$
- Define a functor $F_V : \mathrm{Br}(S_n) \to \mathbb{Z}_2\text{-}\mathbf{gVect}$ with $F_V(\pi) = A_\pi$

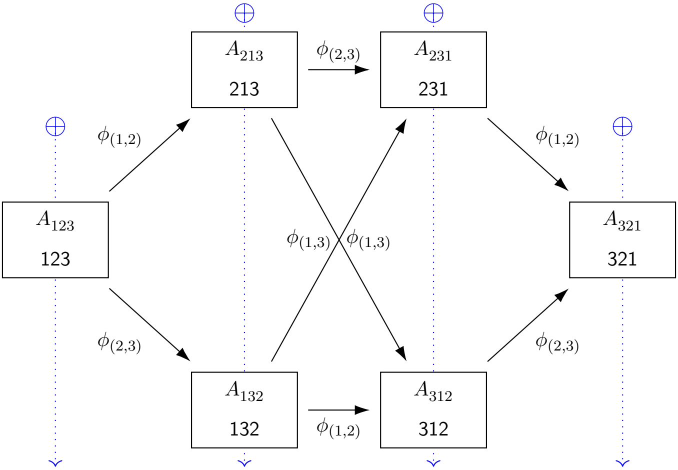
Computing the Vandermonde determinant

E.g. $n = 3$



$$V(\vec{x}) = x_1^{s_1}x_2^{s_2}x_3^{s_3} \quad - x_1^{s_2}x_2^{s_1}x_3^{s_3} - x_1^{s_1}x_2^{s_3}x_3^{s_2} \quad + x_1^{s_2}x_2^{s_3}x_3^{s_1} + x_1^{s_3}x_2^{s_1}x_3^{s_2} \quad - x_1^{s_3}x_2^{s_2}x_3^{s_1}$$

Upgrade to a Functor



$$\left(C^0(\mathrm{Br}(S_n, F_V)) \longrightarrow C^1(\mathrm{Br}(S_n, F_V)) \longrightarrow C^2(\mathrm{Br}(S_n, F_V)) \longrightarrow C^3(\mathrm{Br}(S_n, F_V)) \right)$$

A Categorification of the Vandermonde determinant

Theorem (C., 2016)

For any $\vec{x}, \vec{s} \in \mathbb{Z}_+^n$,

$$\chi\Big(H^*\left(\mathrm{Br}(S_n, F_V)\right)\Big) = V_{\vec{s}}(\vec{x})$$

Remark: Actually this complex can be thought of arising from geometric construction applying families of special 1+1 dimensional TQFTs to smoothings of knot/link diagrams.

Khovanov-like theories

After the success of Khovanov's categorification of the Jones polynomial, many authors used functors on thin posets to categorify other polynomial invariants:

- Helme-Guizon and Rong categorify the chromatic polynomial for graphs
- Sazdanović and Yip categorified the Stanley chromatic symmetric function for graphs
- Stošić categorified the dichromatic polynomial for graphs
- Dansco and Licata categorified the characteristic polynomial of hyperplane arrangements

The Category of Khovanov-Like Theories

Definition (PoCo for Poset Cohomology)

Let **PoCo** denote the category with

- objects (P, F) where P is a thin poset and $F : P \to \mathbb{Z}_2$-**gVect**.
- morphisms $(\phi, \eta) : (P, F) \to (Q, G)$ where $\phi : P \to Q$ preserves cover relations and $\eta : F \to G\phi$ is a natural transformation.

Definition (PoHo for Poset Homology)

Similarly let **PoHo** denote the category with objects (P, F) where $F : P \to \mathbb{Z}_2$-**gVect** is contravariant (i.e. $F : P^{op} \to \mathbb{Z}_2$-**gVect**).

Remark: Later we will discuss how to replace $\mathbb{Z}_2$-**gVect** with any abelian category using "balanced colorings".

Functoriality and Computational Tools

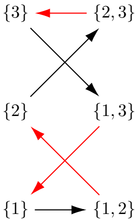
Theorem (C., 2018)

- *H^* is a functor from* **PoCo** *to the category of $\mathbb{Z}_2$-bigraded vector spaces*
- *H_* is a functor from* **PoHo** *to the category of $\mathbb{Z}_2$-bigraded vector spaces*
- *For any upper order ideal U in P there are long exact sequences*

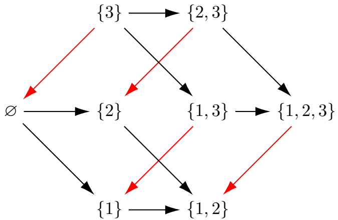
$$
\begin{array}{ccc}
 & H^*(P,F) & \\
\nearrow & & \searrow \\
H^*(U,F) & \xleftarrow[\deg 1]{} & H^*(P \setminus U, F)
\end{array}
\qquad
\begin{array}{ccc}
 & H_*(P,F) & \\
\swarrow & & \nwarrow \\
H_*(U,F) & \xrightarrow[\deg -1]{} & H_*(P \setminus U, F)
\end{array}
$$

Morse matchings in **PoCo** and **PoHo**

- A *complete matching* in a poset P is a collection of disjoint edges in the Hasse diagram
- A complete matching M in P is *Morse* (or *acyclic*) if there are no directed cycles after turning around all arrows in M



not acyclic



acyclic

(matching shown in red)

Simplifying homology calculations with acyclic matchings

Theorem

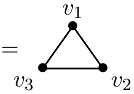
*Given $(P, F) \in$ **PoCo** let $\mathcal{I}$ be an order ideal in P. If $\mathcal{I}$ has an Morse matching M such that $F(e)$ is an isomorphism for each $e \in M$, then $C^*(P, F)$ is homotopy equivalent to $C^*(P \setminus \mathcal{I}, F)$.*

The proof follows almost immediately from the "main theorem" in algebraic Morse theory.

Application: Broken Circuits in Chromatic Homology

- A proper vertex coloring of a graph G is a map $c : V \to [x]$, $x \in \mathbb{N}$, such that no adjacent vertices have the same 'color'
- The *chromatic polynomial* of a graph $G = (V, E)$ is defined as

$$P_G(x) = \#\text{ of proper vertex colorings of } G \text{ with } x \text{ colors}$$

E.g. $G =$ 

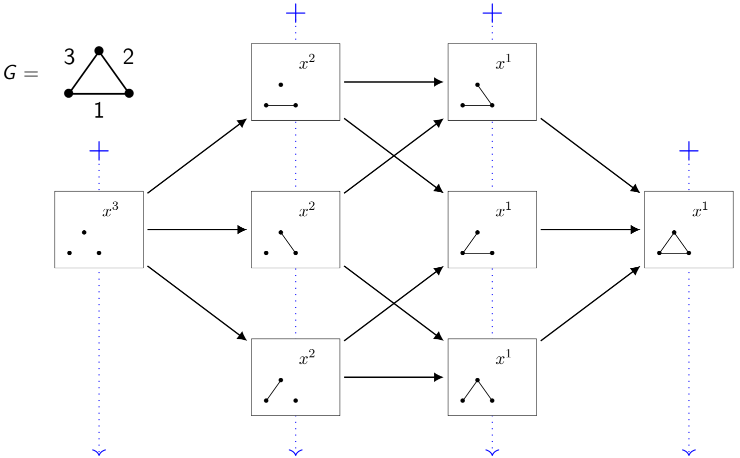
$$\begin{aligned} P_G(x) &= (\#\text{ ways to color } v_1) \\ &\quad \cdot(\#\text{ ways to then color } v_2) \\ &\quad \cdot(\#\text{ ways to then color } v_3) \\ &= x(x-1)(x-2) \end{aligned}$$

- An inclusion-exclusion argument gives the formula

$$P_G(x) = \sum_{S \in 2^E} (-1)^{|S|} x^{k(S)}$$

where $k(S)$ is the number of connected components in S

Computing the chromatic polynomial

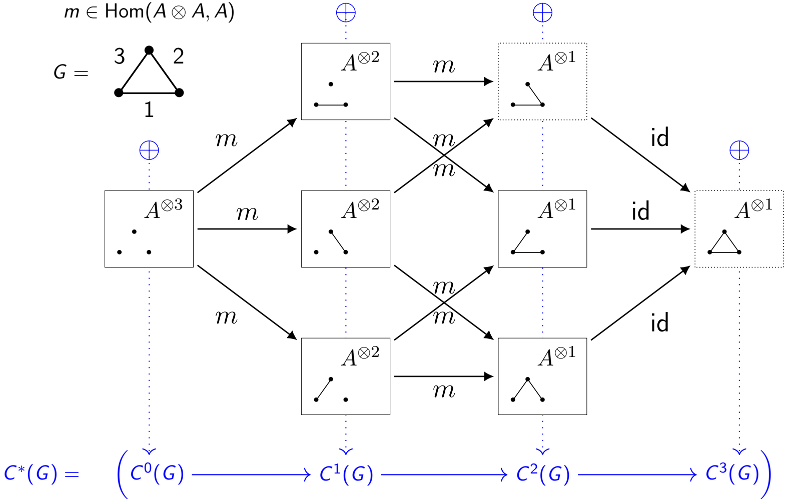


$\chi_G(1+q) = +(1+q)^3 \quad -3(1+q)^2 \quad +3(1+q)^1 \quad -(1+q)^1$

Upgrading to chromatic cohomology

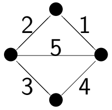
$A \in \mathbb{Z}_2\text{-}\mathbf{gVect}$

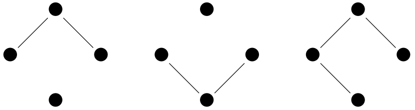
$m \in \mathrm{Hom}(A \otimes A, A)$



Whitney's broken circuit theorem

- Let $G = (V, E)$ and fix an ordering of the edges
- A *broken circuit* in G is $C - e$ where C is a cycle and e is the largest labeled edge of C.
- E.g.

$G =$ 

: 

- Let NBC(G) denote the set of spanning subgraphs of G which do not contain any broken circuits
- Whitney's broken circuit theorem states that

$$P_G(x) = \sum_{S \in 2^E} (-1)^{|S|} x^{k(S)} = \sum_{S \in \mathrm{NBC}(G)} (-1)^{|S|} x^{k(S)}$$

Categorifying Whitney's broken circuit theorem

$$\mathrm{BC}(G) = \{\text{spanning subgraphs which do contain broken circuits}\}$$
$$\mathrm{NBC}(G) = 2^E \setminus \mathrm{BC}(G)$$

Theorem (C., 2018)

For any graph G, there is a complete acyclic matching M on $\mathrm{BC}(G)$ such that each $e \in M$ goes between subgraphs with the same number of connected components.

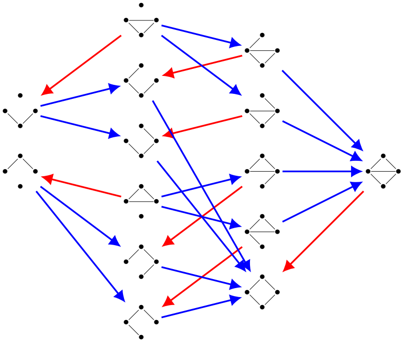
Proof: Given $S \in \mathrm{BC}(G)$ pick e_S maximally such that S contains a broken circuit $C - e_S$. Define an involution $i : \mathrm{BC}(G) \to \mathrm{BC}(G)$ via

$$i(S) = \begin{cases} S + e_S & \text{if } e_S \notin S \\ S - e_S & \text{if } e_S \in S \end{cases}$$

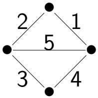
Proof Continued by Example

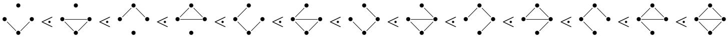
Showing a matching is acyclic is equivalent to finding a linear extension in which each matched edge is a cover relation.

$\mathrm{BC}(G) =$



$G =$





Categorifying Whitney's broken circuit theorem

Given a graph $G = (V, E)$, the chromatic complex $C^*(G) = C^*(2^E, F_{\text{Ch}})$ where $F_{\text{Ch}} : 2^E \to \mathbb{Z}_2$-**gVect** takes $S \subseteq E$ to $A^{\otimes k(S)}$.

Corollary (C., Sazdanovic, Yip 2018)

For any graph G, $C^*(G) = C^*(2^E, F_{\text{Ch}})$ is homotopy equivalent to $C^*(\text{NBC}(G), F_{\text{Ch}})$. The same can be said for Sazdanovic and Yip's categorification of the Stanley chromatic symmetric function.

Thank you!