Two categorifications and a prelim

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Category theory basics

A *category* $\mathcal{C}$ consists of

- a class of objects $\mathrm{Ob}(\mathcal{C})$
- a class $\mathrm{Hom}(x, y)$ of morphisms for each $x, y \in \mathrm{Ob}(\mathcal{C})$
- an associative composition of morphisms
- identity morphisms w.r.t this composition

Categorification (a philosophy)

Categorification is the process of finding category theoretic analogues of set theoretic ideas by 'upgrading':

categorification $\longrightarrow$

sets	categories
elements	objects
equations between elements	isomorphisms between objects
functions	functors

$\longleftarrow$ decategorification

Decategorification is the reverse process (forgetting the extra structure)

Goals of this talk

Discuss

- Categorification of the Vandermonde determinant
- Categorification of the Laguerre polynomials

Example from combinatorics

Objects: finite sets

Morphisms: functions

The category FinSet categorifies $\mathbb{N}$. Decategorify by taking cardinality

- $S \in \operatorname{Ob}\mathrm{FinSet}$ decategorifies to $|S|$
- $S \uplus T$ decategorifies to $|S \uplus T| = |S| + |T|$
- $S \times T$ decategorifies to $|S \times T| = |S|\,|T|$

Combinatorialists use categorification routinely!

E.g. to prove $\binom{n+m}{n} = \sum_{k\geq 0}\binom{n}{k}\binom{m}{k}$, it is easiest to work in the categorified setting. (Notation: let $[n] = \{1, 2, \ldots, n\}$)

- $\binom{n+m}{n}$ is categorified by $\binom{[n]\uplus[m]}{n}$, subsets of $[n]\uplus[m]$ with n elements
- $\sum_{k\geq 0}\binom{n}{k}\binom{m}{k}$ is categorified by $\biguplus_{k\geq 0}\binom{[n]}{k}\times\binom{[m]}{k}$
- There is an bijection $\biguplus_{k\geq 0}\binom{[n]}{k}\times\binom{[m]}{k}\cong\binom{[n]\uplus[m]}{n}$ sending

$$(S, T)\mapsto([n]\setminus S)\cup T$$

- This fact decategorifies to the formula $\binom{n+m}{n} = \sum_{k\geq 0}\binom{n}{k}\binom{m}{k}$

Another categorification of $\mathbb{N}$

Objects:	Morphisms:
f.g. R-modules	R-mod homomorphisms

R comm ring. The category $\underline{R\text{-mod}}$ categorifies $\mathbb{N}$.
Decategorify by taking rank.

- $M \in \operatorname{Ob} R\text{-mod}$ decategorifies to $\operatorname{rank} M$
- $N \oplus M$ decategorifies to $\operatorname{rank} N + \operatorname{rank} M$
- $N \otimes M$ decategorifies to $\operatorname{rank} N \operatorname{rank} M$

Cochain complexes

To get integers we will need *cochain complexes* (C^*, d)

- A *cochain complex* over R-mod is $C^* = \bigoplus_{i\in\mathbb{Z}} C^i$ with
- $C^i \in R$-mod
- a *differential* $d : C^* \to C^*$ with $d(C^i) \subseteq C^{i+1}$
- write $d^i = d|_{C^i}$
- $d^2 = 0$
- has homology $H^i(C^*) = \ker d^i / \operatorname{Im} d^{i-1}$
- has *Euler characteristic* $\chi(C^*) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{rank} C^i = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{rank} H^i(C^*)$

Categorifying $\mathbb{Z}$

$$\underbrace{\text{Objects: bounded cochain complexes} \quad \text{Morphisms: chain maps}}$$

R comm ring. The category $\mathcal{C}^b(R\text{-mod})$ categorifies $\mathbb{Z}$.

- $C^* \in \mathrm{Ob}\,\mathcal{C}^b(R\text{-mod})$ decategorifies to $\chi(C^*)$
- $C^* \oplus D^*$ decategorifies to $\chi(C^*) + \chi(D^*)$
- $C^* \otimes D^*$ decategorifies to $\chi(C^*)\chi(D^*)$

Categorifying linear algebra

- Categorical diagonalization (Elias and Hogankamp)
- Categorification of the trace of the Coxeter matrix (Happel)
- Categorified determinants?

Goal #1: Categorify the Vandermonde determinant

The **Vandermonde determinant** is defined as

$$V_n = \begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{\pi(1)} x_2^{\pi(2)} \ldots x_n^{\pi(n)}$$

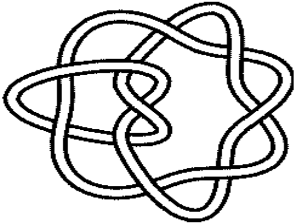
- $V_n = x_1 \ldots x_n \prod_{i<j}(x_j - x_i)$
- Every alternating polynomial is a multiple of $\tilde{V}_n = \frac{V_n}{x_1 \ldots x_n}$
- Appears in Frobenius formula which calculates characters of representations of S_n

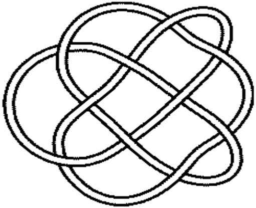
The Plan

- Categorify (evaluations of) V_n for $x_1, \dots, x_n \in \mathbb{N}$
- Accomplish this in a way analogous to Khovanov's categorification of the Jones Polynomial
- First review Khovanov homology

Basic Question in Topology

Are these knots the same? Not easy to say.

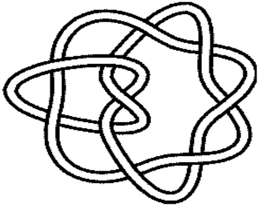




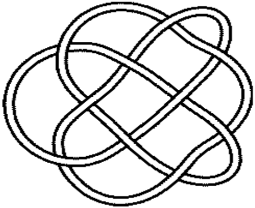
To answer such questions we use *knot invariants*

Using knot invariants to distinguish knots

The Jones polynomial is good at telling knots apart:



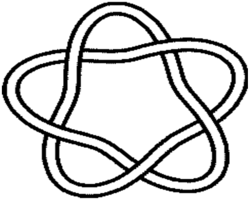
$$J(q) = -q^4 + q^3 - q^2 + 2q - 1 + 2q^{-1} - q^{-2} + q^{-3} - q^{-4}$$



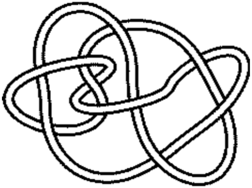
$$J(q) = q^3 - 4q^2 + 8q - 11 + 15q^{-1} + 15q^{-3} - 12q^{-4} + 8q^{-5} - 4q^{-6} + q^{-7}$$

Polynomials are different $\implies$ knots are different

When Jones doesn't do the trick



$J(q) = 2q^{-1} - 3q^{-2} + 5q^{-3}$
$-5q^{-4} + 5q^{-5} - 5q^{-6}$
$+3q^{-7} - 2q^{-8} + q^{-9}$



$J(q) = 2q^{-1} - 3q^{-2} + 5q^{-3}$
$-5q^{-4} + 5q^{-5} - 5q^{-6}$
$+3q^{-7} - 2q^{-8} + q^{-9}$

Polynomials equal $\not\Rightarrow$ knots are the same

Khovanov Homology

Categorifies the Jones polynomial

$$J(K) = \sum_{\alpha \in \{0,1\}^n} (-1)^{h(\alpha)} (q + q^{-1})^{s(\alpha)} \in \mathbb{Z}[q, q^{-1}]$$

where K is a knot diagram with n crossings,

$$h(\alpha) = \#1's \text{ in } \alpha,$$

$$s(\alpha) = \#\text{circles in } \alpha\text{-smoothing}$$

Categorifying $\mathbb{N}[q, q^{-1}]$

Objects: graded R-modules

Morphisms: graded homomorphisms

R comm ring. The category R-gmod categorifies $\mathbb{N}[q, q^{-1}]$.
The *graded rank* of $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is $\operatorname{qrank} M = \sum_{n\in\mathbb{Z}} q^n \operatorname{rank} M_n$.

- $M = \bigoplus_{n\in\mathbb{Z}} M_n$ decategorifies to $\operatorname{qrank} M$
- $M \oplus N$ decategorifies to $\operatorname{qrank} M + \operatorname{qrank} N$
- $M \otimes N$ decategorifies to $\operatorname{qrank} M \operatorname{qrank} N$

Categorifying $\mathbb{Z}[q, q^{-1}]$

Objects: bounded cochain complexes of graded R-modules

Morphisms: graded chain maps

R comm ring. The category $\mathcal{C}(R\text{-gmod})$ categorifies $\mathbb{Z}[q, q^{-1}]$.
The *graded Euler characteristic* is $\chi_q(C^*) = \sum_{r \in \mathbb{Z}} (-1)^r \operatorname{qrank} C^r$

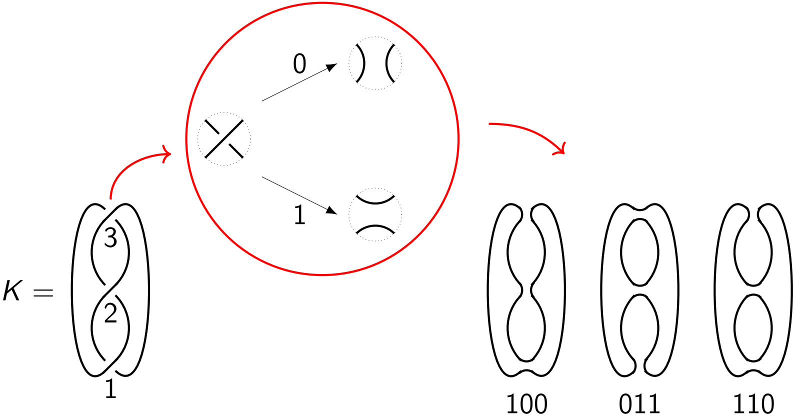
- $C^* \in \operatorname{Ob} \mathcal{C}(R\text{-gmod})$ decategorifies to $\chi_q(C^*)$
- $C^* \oplus D^*$ decategorifies to $\chi_q(C^*) + \chi_q(D^*)$
- $C^* \otimes D^*$ decategorifies to $\chi_q(C^*)\chi_q(D^*)$

The n-cube C_n

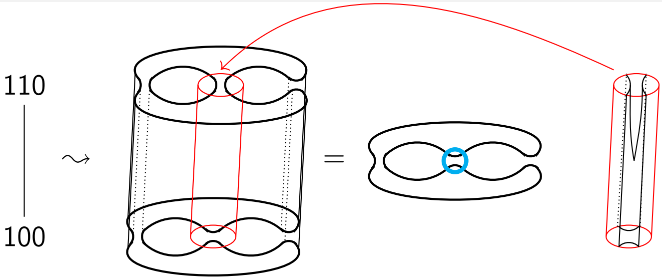
- Partially ordered set with vertices $\{0,1\}^n$
- Cover relation (edge) when you change a 0 to a 1:

$$C_3 : \quad \begin{tikzcd}[arrows=-]
 & 100 \arrow[r] \arrow[dr] & 110 \arrow[dr] & \\
000 \arrow[ur] \arrow[r] \arrow[dr] & 010 \arrow[ur] \arrow[dr] & 101 \arrow[r] & 111 \\
 & 001 \arrow[ur] \arrow[r] & 011 \arrow[ur] &
\end{tikzcd}$$

n-tuples of 1's and 0's encode **smoothings** of K



Edges encode **cobordisms** between smoothings



- Start with (100-smoothing) $\times [0, 1]$
- Remove cylindrical neighborhood of changing crossing
- Replace with a saddle

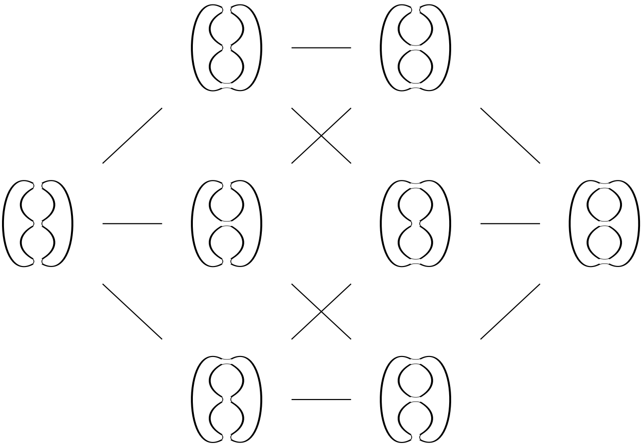
Category of cobordisms

Objects: closed 1D manifolds

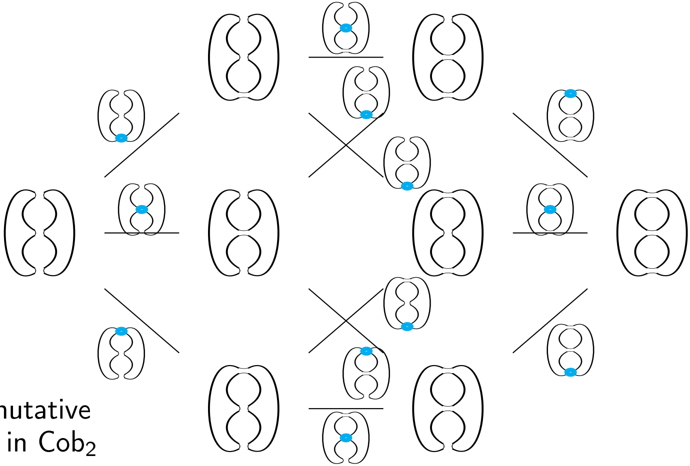
Morphisms: 2D surfaces with boundary

The category Cob_2 contains smoothings and cobordisms as its objects and morphisms

Replace vertices in C_n by corresponding smoothings



Replace edges in C_n by cobordisms

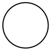
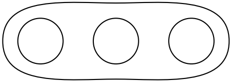


A commutative diagram in Cob_2

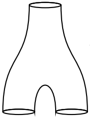
2D TQFTs and Frobenius algebras

A 2D TQFT is a monoidal functor from Cob_2 to R-mod.

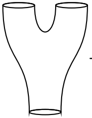
- assigns an R-module A to each circle

 $\longrightarrow$ A  $\longrightarrow$ $A \otimes A \otimes A \otimes A$

- assigns linear maps to cobordisms

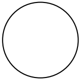
 $\longrightarrow \left(m : A \otimes A \to A \right)$ multiplication

 $\longrightarrow \left(\eta : R \to A \right)$ unit

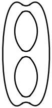
 $\longrightarrow \left(\Delta : A \to A \otimes A \right)$ comultiplication

 $\longrightarrow \left(\epsilon : A \to R \right)$ counit

Pick $A \in R\text{-gmod}$ such that $\operatorname{qrank} A = q + q^{-1}$



$$\begin{tikzcd} & A \arrow[d, "\text{qrank}"] \\ {} \arrow[ur, "TQFT"] \arrow[r, "Jones"'] & q + q^{-1} \end{tikzcd}$$



$$\begin{tikzcd} & A \otimes A \otimes A \arrow[d, "\text{qrank}"] \\ {} \arrow[ur, "TQFT"] \arrow[r, "Jones"'] & (q + q^{-1})^3 \end{tikzcd}$$

Apply a 2D TQFT

$$\begin{tikzcd}
 & A \arrow[r, "\Delta"] \arrow[dr, "\Delta"] & A \otimes A \arrow[dr, "\mathrm{Id} \otimes \Delta"] & \\
A \otimes A \arrow[ur, "m"] \arrow[r, "m"] \arrow[dr, "m"] & A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"] & A \otimes A \arrow[r, "\mathrm{Id} \otimes \Delta"] & A \otimes A \otimes A \\
 & A \arrow[ur, "\Delta"] \arrow[r, "\Delta"] & A \otimes A \arrow[ur, "\mathrm{Id} \otimes \Delta"] &
\end{tikzcd}$$

An anti-commutative diagram in R-gmod

Direct sum down ranks and get a chain complex

$$
\begin{tikzcd}
 & A \arrow[r, "\Delta"] \arrow[dr, "\Delta"] & A \otimes A \arrow[dr, "\mathrm{Id} \otimes \Delta"] & \\
A \otimes A \arrow[ur, "m"] \arrow[r, "m"] \arrow[dr, "m"'] \arrow[dd, dotted] & A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"] & A \otimes A \arrow[r, "\mathrm{Id} \otimes \Delta"] & A \otimes A \otimes A \arrow[dd, dotted] \\
 & A \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] \arrow[d, dotted] & A \otimes A \arrow[ur, "\mathrm{Id} \otimes \Delta"'] \arrow[d, dotted] & \\
A \otimes A \arrow[r] & A^{\oplus 3} \arrow[r] & (A \otimes A)^{\oplus 3} \arrow[r] & A \otimes A \otimes A
\end{tikzcd}
$$

Theorem (Khovanov)

The homology groups of this chain complex are knot invariants and the graded Euler characteristic of this complex is equal to the Jones polynomial

$$J(K) = \sum_{i \in \mathbb{Z}} (-1)^i q\mathrm{rank}\, H^i$$

- The Khovanov homology groups give a strictly stronger knot invariant than the Jones polynomial
- Khovanov homology is a functor. That is, cobordisms between knots induce maps between Khovanov homologies

Why did this construction work?

- The Jones polynomial is a rank alternating sum over a ranked poset $P = C_n$

$$\sum_{x \in P} (-1)^{r(x)} f(x)$$

- Every interval of length 2 in C_n is a diamond (i.e. C_n is **thin**)
- There is a $\{+1, -1\}$ edge coloring of C_n for which each diamond has an odd number of -1's (a **balanced coloring**)

Categorifying Vandermonde

$$V_n = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{\pi(1)} x_2^{\pi(2)} \dots x_n^{\pi(n)}$$

- S_n has a thin poset structure: the **Bruhat order**
- $\mathrm{inv}(\pi)$ is the rank function for this ordering
- The Bruhat order has a balanced coloring

We're in business!

Bruhat Order on S_n

- An **inversion** in a permutation π is a pair (i, j) with $i < j$ and $\pi(i) > \pi(j)$
- $\text{inv}(\pi)$ denotes the number of inversions of π
  E.g. $\text{inv}(21534) = 3$

$$2\ 1\ 5\ 3\ 4 \qquad 2\ 1\ 5\ 3\ 4 \qquad 2\ 1\ 5\ 3\ 4$$

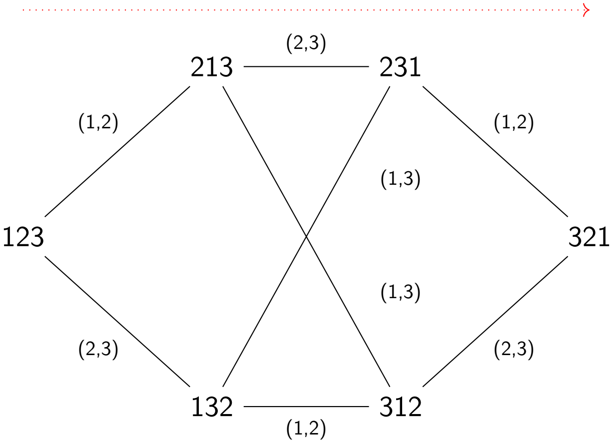
Bruhat Order on S_n

- Has a vertex for each $\pi \in S_n$ (one line notation)
- Has an edge (cover relation) $\pi \lessdot \sigma$ whenever σ is gotten from π by transposing a non-inversion pair for which $\mathrm{inv}(\sigma) = \mathrm{inv}(\pi) + 1$
- E.g.

$$2\ \underbrace{1\ 5\ 3}\ 4 \quad \lessdot \quad 2\ 3\ 5\ 1\ 4$$
goes up by one inversion

$$\underbrace{2\ 1\ 5\ 3\ 4} \quad \not\lessdot \quad 4\ 1\ 5\ 3\ 2$$
goes up by two inversions

E.g. Bruhat Order on S_3



Colored Cobordisms: Cob_2^n

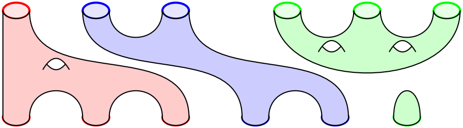
Objects: $[n]$-colored closed 1-manifolds

Morphisms: color preserving cobordisms

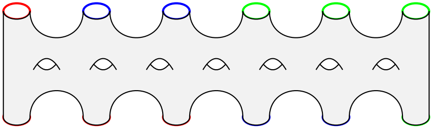
Let $[n] = \{1, 2, \ldots, n\}$. The category Cob_2^n has

- Objects: closed 1-manifolds with each connected component given a color from $[n]$
- Morphisms: 2-dimensional manifolds for which each connected component has monochromatic boundary

For example, let $M =$ ○○○○○○ and $N =$ ○○○○○○



is a colored cobordism from M to N, but not



Permutations (vertices) encode colored smoothings

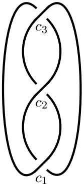
K a knot diagram with crossings $c_1, \ldots, c_n$ and $\pi \in S_n$

- The π-**smoothing** of K is

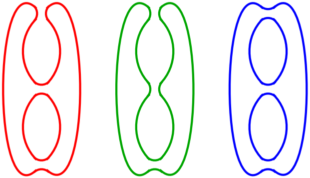
$$K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \text{Ob Cob}_2^n$$

- K_i^π is gotten by giving $c_1, c_2, \ldots, c_{\pi(i)}$ 1-smoothings and all other 0-smoothings
- All components of K_i^π are colored i

π-Smoothing Example

$K =$ 

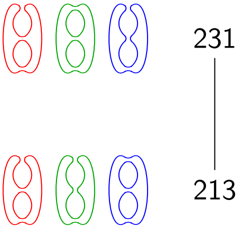
$\xrightarrow{\pi = 213 \text{ smoothing}}$



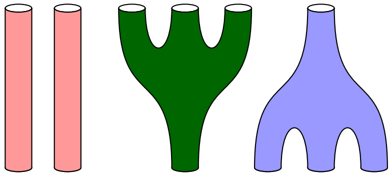
K_1^π K_2^π K_3^π

Edges encode colored cobordisms

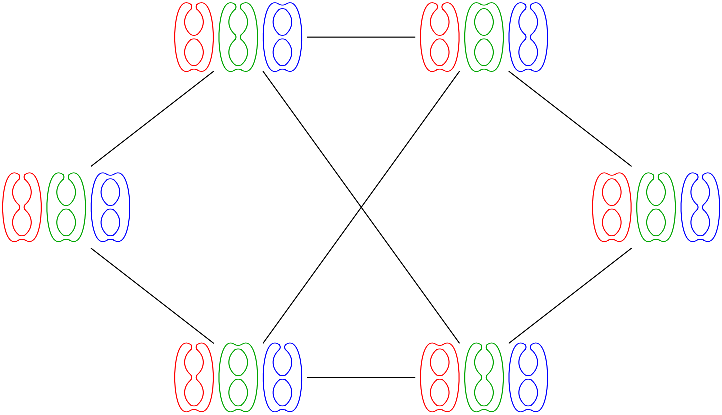
If $\pi \lessdot \sigma$ then $K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \mathrm{Ob}\ \mathrm{Cob}_2^n$ and
$K^\sigma = K_1^\sigma \amalg K_2^\sigma \amalg \ldots \amalg K_n^\sigma \in \mathrm{Ob}\ \mathrm{Cob}_2^n$
differ at exactly two colors. Use connected genus 0 cobordisms on the colored pieces which differ, and identity (cylinders) on pieces which do not change



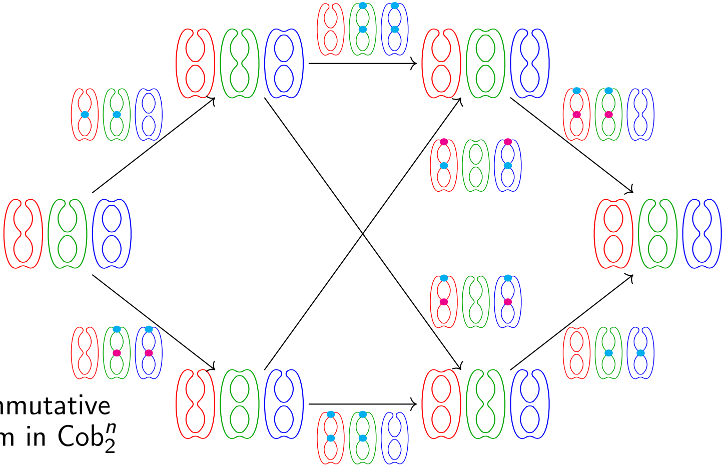
$\rightsquigarrow$



Replace vertex π with π-smoothing



Replace edges with colored cobordisms

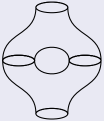
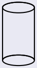


A commutative diagram in Cob_2^n

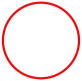
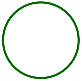
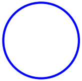
2D special colored TQFTs

Definition

- A 2D TQFT F is **special** if the following condition holds:

$F\Big($  $\Big) = F\Big($  $\Big) \quad \Longleftrightarrow \quad \mu \circ \Delta = 1$

- A **special colored TQFT** is a monoidal functor $F : \mathrm{Cob}_2^n \to R\text{-mod}$ which restricts to a special TQFT on each color.

 $\rightsquigarrow A_{x_1}$, rank $= x_1$  $\rightsquigarrow A_{x_2}$, rank $= x_2$ $\cdots$  $\rightsquigarrow A_{x_n}$, rank $= x_n$

Apply a Special Colored TQFT

$$
\begin{tikzcd}
 & A_{x_1}^{\otimes 2}\otimes A_{x_2}\otimes A_{x_3}^{\otimes 3} \arrow[r, "\mathrm{Id}\otimes\Delta^2\otimes m^2"] \arrow[dr, "\Delta^2 m\otimes\mathrm{Id}\otimes m^2\Delta"] & A_{x_1}^{\otimes 2}\otimes A_{x_3}^{\otimes 3}\otimes A_{x_3} \arrow[dr, "\Delta^2 m\otimes\Delta m^2\otimes\mathrm{Id}"] & \\
A_{x_1}\otimes A_{x_2}^{\otimes 2}\otimes A_{x_3}^{\otimes 3} \arrow[ur, "\Delta\otimes m\otimes\mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes\Delta^2 m\otimes\Delta m^2"'] & & & A_{x_1}^{\otimes 3}\otimes A_{x_2}^{\otimes 2}\otimes A_{x_3} \\
 & A_{x_1}\otimes A_{x_2}^{\otimes 3}\otimes A_{x_3}^{\otimes 2} \arrow[r, "\Delta^2\otimes m^2\otimes\mathrm{Id}"'] \arrow[ur, "\Delta\otimes\mathrm{Id}\otimes m"] & A_{x_1}^{\otimes 3}\otimes A_{x_2}\otimes A_{x_3}^{\otimes 2} \arrow[ur, "\mathrm{Id}\otimes\Delta\otimes m"'] &
\end{tikzcd}
$$

An anti-commutative diagram in R-mod

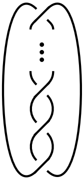
Direct sum down ranks to get a chain complex

$$
\begin{tikzcd}
 & A_{x_1}^{\otimes 2}\otimes A_{x_2}\otimes A_{x_3}^{\otimes 3} \arrow[rr, "\mathrm{Id}\otimes\Delta^2\otimes m^2"] \arrow[ddrr, "\Delta^2 m\otimes \mathrm{Id}\otimes m^2\Delta" near end] & & A_{x_1}^{\otimes 2}\otimes A_{x_3}^{\otimes 3}\otimes A_{x_3} \arrow[dr, "\Delta^2 m\otimes\Delta m^2\otimes \mathrm{Id}"] & \\
A_{x_1}\otimes A_{x_2}^{\otimes 2}\otimes A_{x_3}^{\otimes 3} \arrow[ur, "\Delta\otimes m\otimes \mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes\Delta^2 m\otimes\Delta m^2"'] & & & & A_{x_1}^{\otimes 3}\otimes A_{x_2}^{\otimes 2}\otimes A_{x_3} \\
 & A_{x_1}\otimes A_{x_2}^{\otimes 3}\otimes A_{x_3}^{\otimes 2} \arrow[rr, "\Delta^2\otimes m^2\otimes \mathrm{Id}"'] \arrow[uurr, "\Delta\otimes \mathrm{Id}\otimes m" near end] & & A_{x_1}^{\otimes 3}\otimes A_{x_2}\otimes A_{x_3}^{\otimes 2} \arrow[ur, "\mathrm{Id}\otimes\Delta\otimes m"'] &
\end{tikzcd}
$$

Theorem (C.)

Let K be the alternating two strand braid diagram of the $(2, n)$-torus knot. Then the Euler characteristic of this chain complex is equal to the Vandermonde determinant

$$V_n = \sum_{i \geq 0} (-1)^i dim H^i$$

$K =$ 

What's Next?

Questions:

- Is this categorification functorial?
- This process also works for generalized Vandermonde determinants. Can this be used to categorify the Frobenius formula?
- Relation to $V_n = x_1 \dots x_n \prod_{i<j}(x_j - x_i)$

Goal #2: Categorify the Laguerre polynomials

Orthogonal Polynomials

The polynomial ring $\mathbb{Q}[x]$ has standard basis $\{1, x, x^2, \ldots\}$ orthogonal with respect to bilinear form

$$(f, g) = \sum_{n \geq 0} \left([x^n]f\right) \left([x^n]g\right)$$

By changing the bilinear form we can get other families of orthogonal polynomials.

Orthogonal Polynomials

We consider the bilinear form

$$(f, g) = \int_a^b f(x)g(x)w(x)\mathrm{d}x$$

for some choice of weight function $w(x)$ and interval $[a, b]$.
Apply Gram-Schmidt to the standard basis $\{1, x, x^2, \dots\}$ w.r.t. this bilinear form to get a family of orthogonal polynomials.

$w(x)$	e^{-x^2}	$\sqrt{1-x^2}$	e^{-x}
(a, b)	$(-\infty, \infty)$	$(-1, 1)$	$(0, \infty)$
Family	Hermite	Chebyshev	Laguerre

Laguerre Polynomials

Consider the choice:

$$(f, g) = \int_0^\infty f(x)g(x)e^{-x}\mathrm{d}x.$$

Applying Gram-Schmidt we get the Laguerre polynomials

$$L_n(x) = \sum_{k\geq 0}(-1)^k \binom{n}{k}\frac{x^k}{k!}$$

The Plan

- Categorify $L_n(x)$ similarly to Khovanov and Sazdanovic's categorification of Hermite and Chebyshev polynomials
- First review their categorifications

Grothendieck Groups

has 'direct sums'

Def: The Grothendieck group $K_0(\mathcal{A})$ of (additive) category $\mathcal{A}$

- Free Abel. group generated by symbols $[M]$ for $M \in \mathrm{Ob}(\mathcal{A})$
- Subject to relations $[M] = [N] + [K]$ whenever $M \cong N \oplus K$
- For example $K_0(\mathbb{k}\text{-Vect}_{fin}) \cong \mathbb{Z}$ generated by $[\mathbb{k}]$

Categorification of abelian groups

To categorify some element g in an abelian group G:

- Construct a category $\mathcal{C}$ with $K_0(\mathcal{C}) \cong G$
- Find an object $M_g \in \mathrm{Ob}(\mathcal{C})$ such that $[M_g] \mapsto g$ under this isomorphism (make identification $[M_g] = g$)

Question: What can we learn about g by looking at M_g?

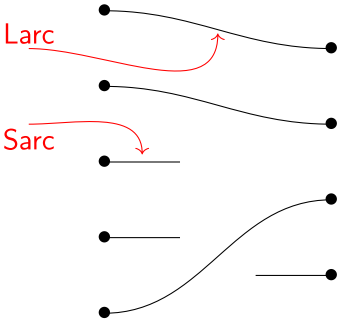
Categorifying a Polynomial $f(x) \in \mathbb{Z}[x]$

- Category $\mathcal{C}$ with $K_0(\mathcal{C}) \cong \mathbb{Z}[x]$
- Family of objects P_n with $[P_n] = x^n$
- Object M with $[M] = f(x)$

In 2011, Sazdanovic and Khovanov constructed such a category $\mathcal{C}$ via the “slarc” algebra.

Def An (n, m)-slarc diagram has

- n left endpoints and m right endpoints
- Each endpoint has a short arc (sarc) or a long arc (larc)
- No crossings or returns



$(5, 4)$-slarc diagram

Width=3

Def Let ${}_nB_m$ denote the set of (n, m)-slarc diagrams. Then

$$|{}_nB_m| = \sum_{k\geq 0} \binom{n}{k}\binom{m}{k} = \binom{n+m}{n}$$

width

Define

$$B = \biguplus_{n,m\geq 0} {}_nB_m$$

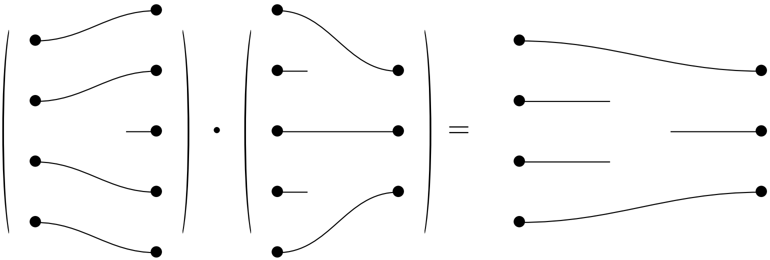
and define A as the $\Bbbk$-vector space with basis B:

$$A = \Bbbk B$$

Algebra of slarcs

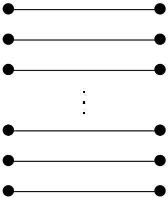
Algebra structure on $A = \Bbbk B$:

- Multiply compatible diagrams by horizontal concatenation
- Incompatible diagrams have product 0
- Floating arcs are set to 0:  $= 0$



Orthogonal idempotents

Def: Let $1_n \in {}_nB_n$ be the unique diagram with no sarcs.

$1_n =$ 

The elements $\{1_n \mid n \in \mathbb{N}\}$ are mutually orthogonal idempotents:

$$1_n 1_n = 1_n$$

$$1_n 1_m = 0 \quad (n \neq m)$$

- **Def:** Let M be an A-module. M is projective if $M \oplus N$ is free for some A-module N.
- Let A-pmod denote the category of f.g. projective A-modules.
- The modules $P_n := A \cdot 1_n$ are indecomposable projectives
- $\mathrm{Hom}(P_n, P_m) = \Bbbk \cdot {}_nB_m$

Theorem (Sazdanovic, Khovanov, 2011)

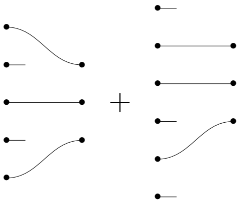
Every f.g. projective A-module can be written as a direct sum of P_n's.

Theorem (Sazdanovic, Khovanov, 2011)

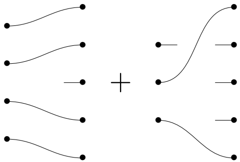
The map $K_0(A\text{-pmod}) \to \mathbb{Z}[x]$ sending $[P_n] \mapsto x^n$ is a group isomorphism.

Standard modules

- Let $K_n \leq P_n$ be the submodule generated by diagrams with at least one right sarc
- Define the **standard module** $M_n = {}^{P_n}\!/_{K_n}$
- M_n has a basis of diagrams with n right endpoints and no right sarcs



Standard M_3

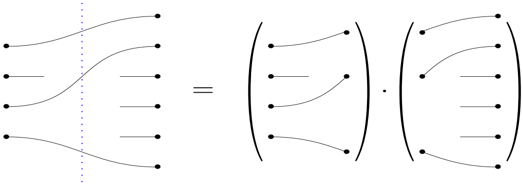


Projective P_5

Relations between P_n and M_n

- Let $P_n(\leq k)$ be the submodule of P_n generated by diagrams of width $\leq k$ (diagrams with $\leq k$ larks)
- P_n has a filtration $P_n(\leq 0) \subseteq P_n(\leq 1) \subseteq \cdots \subseteq P_n(\leq n) = P_n$
- $P_n(\leq k)/P_n(\leq k-1) \cong M_k^{\binom{n}{k}}$

e.g.
$n = 6, k = 3$



Objects: bounded complexes of projectives

Morphisms: chain maps up to chain homotopy

Theorem (Sazdanovic, Khovanov, 2011)

$$K_0(A\text{-}pmod) \cong K_0(\mathcal{C}^b(A\text{-}pmod))$$

via the Euler characteristic map

$$[C^*] \mapsto \sum_{n\in\mathbb{Z}} (-1)^n [C^n].$$

Theorem (Relations in $K_0(A\text{-mod})$)

Given a filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F$ with $F_i/F_{i-1} \cong Q_i$ it follows that

$$[F] = \sum_{i=0}^{n} [Q_i]$$

Given a resolution $0 \to F_0 \to \cdots \to F_n \to F$ it follows that

$$[F] = \sum_{i=0}^{n} (-1)^i [F_i]$$

Relations between P_n and M_n

- Consequently $[P_n] = \sum_{k=0}^{n} \binom{n}{k} [M_k]$ in $K_0(A\text{-mod})$
- Inverting this yields

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k]$$

- Suggests M_n has a resolution by P_k's:

$$0 \to P_0 \to \cdots \to P_k^{\binom{n}{k}} \to \cdots \to P_{n-1}^{\binom{n}{n-1}} \to P_n \to M_n$$

- Sazdanovic and Khovanov constructed such a resolution in 2011

Under the isomorphism $[P_n] \mapsto x^n$ we find that

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} x^k = (x-1)^n$$

so M_n categorifies $(x-1)^n$.

$$\begin{array}{ccc}
0 \to P_0 \to \cdots \to P_{n-1}^{\binom{n}{n-1}} \to P_n \to M_n & & P_n(\leq 0) \subseteq \cdots \subseteq P_n(\leq n) = P_n \\
\Big\downarrow K_0 & & \Big\downarrow K_0 \\
[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k] & & [P_n] = \sum_{k=0}^{n} \binom{n}{k} [M_k] \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
(x-1)^n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} x^k & & x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k
\end{array}$$

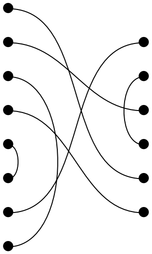
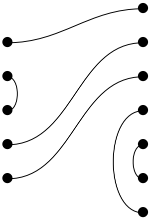
Categorifying Orthogonal Polynomials in $\mathbb{Z}[x]$

Idea: Categorify the bilinear form $(\cdot, \cdot)$ on $\mathbb{Z}[x]$. We need:

- Diagrammatic algebra A
- $K_0(A\text{-pmod}) \cong \mathbb{Z}[x]$ as groups via $[P_n] \mapsto x^n$
- $(x^n, x^m) = \dim \operatorname{Hom}(P_n, P_m) = |{}_nB_m|$

That is, construct a diagrammatic algebra where ${}_nB_m$ is counted by (x^n, x^m).

Categorifying Hermite and Chebyshev Polynomials

	Hermite	Chebyshev
Formula	$\sum(-1)^k\binom{n}{n-2k}(2k-1)!!x^{n-2k}$	$\sum(-1)^k\binom{n-k}{k}x^{n-2k}$
(x^n, x^m)	$(n+m-1)!!$	Catalan C_k, $k=\frac{n+m}{2}$
Diagrams	Allow returns, crossings, no sarcs 	non-crossing matchings 

Categorifying Laguerre Polynomials

$$L_n(x) = \sum (-1)^k \binom{n}{k} \frac{x^k}{k!}$$

- Problem! Coeff of x^k in $L_n(x)$ is not an integer...
- A solution: View $L_n(x)$ as an *exponential polynomial*

$$L_n(x) \in \mathbb{Z}^{\exp}[x] := \mathbb{Z}\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\right]$$

- coeff of $\frac{x^k}{k!}$ in $L_n(x)$ is $(-1)^k \binom{n}{k}$.

Categorifying Laguerre Polynomials

Strategy: Find a diagrammatic algebra A with

- $K_0(A\text{-pmod}) \cong \mathbb{Z}^{\exp}[x]$, $[P_n] \mapsto \frac{x^n}{n!}$
  (easy: since $\mathbb{Z}[x] \cong \mathbb{Z}^{\exp}[x]$ as groups, A-pmod works
- $(\frac{x^n}{n!}, \frac{x^m}{m!}) = \dim \mathrm{Hom}(P_n, P_m)$
- Find modules M_n for which $[M_n] = L_n(x)$

Choosing Diagrammatics

- Bilinear form for Laguerre is $(f, g) = \int_0^\infty f(x)g(x)e^{-x}\mathrm{d}x$
- $(\frac{x^n}{n!}, \frac{x^m}{m!}) = \int_0^\infty \frac{x^n}{n!}\frac{x^m}{m!}e^{-x}\mathrm{d}x = \frac{(n+m)!}{n!m!} = \binom{n+m}{n}$
- recall for slarc algebra A, $\dim \mathrm{Hom}(P_n, P_m) = \binom{n+m}{n}$

Choosing Modules

Recall the standard modules M_n satisfy

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k]$$

Define the *signed standard modules*

$$N_n = M_n[n] = \left(0 \to (M_n)_{(n)} \to 0\right)$$

so in the Grothendieck group

$$[N_n] = (-1)^n [M_n]$$

Choosing Modules

Theorem (C. 2017)

$K_0(A\text{-pmod}) \cong \mathbb{Z}^{exp}[x]$ via $[P_n] \mapsto \frac{x^n}{n!}$ and signed standard modules categorify Laguerre polynomials under this isomorphism

$$[M_n] = (-1)^n L_n(x) \qquad [N_n] = L_n(x)$$

$$\begin{array}{c} 0 \to P_0 \to \cdots \to P_{n-1}^{\binom{n}{n-1}} \to P_n \to M_n \\ \downarrow K_0 \\ [M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k] \\ \downarrow \cong \\ (-1)^n L_n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} \end{array}$$

$$\begin{array}{c} P_n(\le 0) \subseteq \cdots \subseteq P_n(\le n) = P_n \\ \downarrow K_0 \\ [P_n] = \sum_{k=0}^{n} \binom{n}{k} [M_k] \\ \downarrow \cong \\ \frac{x^n}{n!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_k(x) \end{array}$$

Future goal: categorify product structure on $\mathbb{Z}^{\exp}[x]$

- Categorify product $\frac{x^n}{n!} \cdot \frac{x^m}{m!} = \binom{n+m}{n} \frac{x^{n+m}}{(n+m)!}$
- Need a functor $\otimes : A\text{-pmod} \times A\text{-pmod} \to A\text{-pmod}$
- Set $P_n \otimes P_m = P_{n+m}^{\binom{n+m}{n}}$
- Need to describe action on morphisms. That is, given:

$$f : P_n \to P_k$$
$$g : P_m \to P_\ell,$$

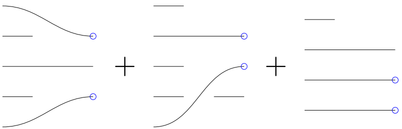
we need to define

$$f \otimes g : P_n \otimes P_m \to P_k \otimes P_\ell$$
$$f \otimes g : P_{n+m}^{\binom{n+m}{n}} \to P_{k+\ell}^{\binom{k+\ell}{k}}$$

Product on A-pmod

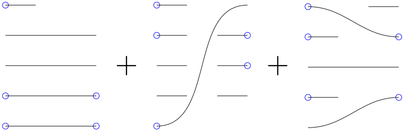
- Identify elements of $P_{n+m}^{\binom{n+m}{n}}$ as formal sums

E.g.
$n=2, m=1$



- Identify maps of $P_{n+m}^{\binom{n+m}{n}} \to P_{k+\ell}^{\binom{k+\ell}{k}}$ as formal sums

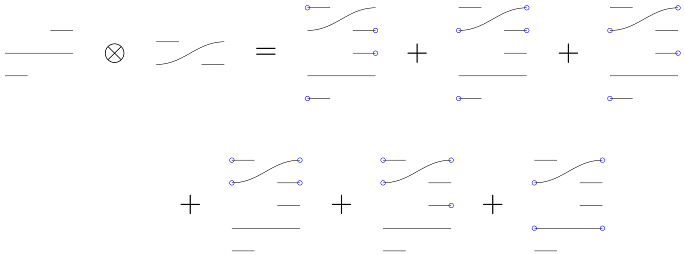
E.g.
$n=3, m=2$
$k=2, \ell=2$



Product on A-pmod

Define product $A\text{-pmod} \times A\text{-pmod} \to A\text{-pmod}$ on morphisms by

1. Stacking diagrams
2. Summing over all labelings which are ‘sub-diagrams’



Future goal: categorify products of Laguerre polynomials

$$L_r(x)L_s(x) = \sum_{t=|r-s|}^{r+s} (-1)^{r+s+t} c_{rst} L_t(x)$$

where $c_{rst} = \sum_n 2^{2n-p} \frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!}$, and $p = r+s+t$

$$\begin{array}{c}
P(M_r) \otimes P(M_s) = F_{r+s} \supset F_{r+s-1} \supset \cdots \supset F_{|r-s|} \\
\Big\downarrow K_0 \\
[M_r] \cdot [M_s] = \sum_{t=|r-s|}^{r+s} c_{rst} \, [M_t] \\
\Big\downarrow \cong \\
(-1)^r L_r(x) \cdot (-1)^s L_s(x) = \sum_{t=|r-s|}^{r+s} c_{rst} (-1)^t L_t(x)
\end{array}$$

Other Projects

- Torsion in Khovanov homology of 3-braids with Victor Summers
- Chromatic homology for signed graphs and its relation to Khovanov homology
- Chapter in MSRI book on categorical diagonalization

Thank you!