# Torsion in the Khovanov homology of 3-strand torus links

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# Goals and Plan

Goal: Prove the conjecture of Sazdanović and Przytycki (2012) that 3-braids have only $\mathbb{Z}_2$ torsion.

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Plan of this talk:

- Recall computations of Turner and Mounir for Khovanov homology of 3-strand torus links over $\mathbb{Q}$ and $\mathbb{Z}_2$
- Prove that 3-strand torus links have only $\mathbb{Z}_2$ torsion and give an explicit formula for their Khovanov homology over $\mathbb{Z}$
- Discuss how this may be extended to (possibly) all 3-braids.

# Khovanov Homology

Given a link $L$, and an abelian group $G$, the Khovanov homology of $L$ with coefficients in $G$ is a bigraded abelian group $\mathrm{Kh}^{**}(L, G)$ whose graded Euler characteristic is the (normalized) Jones polynomial:

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \operatorname{rank} \mathrm{Kh}^{i,j}(L) = \tilde{J}(L).$$

- $\mathrm{Kh}^{**}(L)$ denotes $\mathrm{Kh}^{**}(L, \mathbb{Z})$
- Each $\mathrm{Kh}^{i,j}(L, G)$ is a link invariant
- $\mathrm{Kh}^{**}(L)$ is a stronger link invariant than $\tilde{J}(L)$

# Torsion in Khovanov homology

- $\mathbb{Z}_2$-torsion is abundant in Khovanov homology (experimentally)
- Odd torsion is rare in Khovanov homology (experimentally)
- Shumakovitch (2012) proved $\mathbb{Z}_2$H-thin links have only $\mathbb{Z}_2$-torsion
- In the previous talk, we saw that 3-braids have no odd torsion
- In this talk, we repeat Shumakovitch's argument in the context of 3-strand torus links to show there is only $\mathbb{Z}_2$-torsion

# Khovanov homology of torus links

- In 2006 Turner computed $\mathrm{Kh}^{**}(T_{3,q}, \mathbb{Q})$ for all $q$ (also done independently by Stošić in 2007)
- Actually this computation works just as well over $\mathbb{Z}_p$ for $p$ an odd prime, so for all $i, j$ we have

$$\dim_{\mathbb{Q}} \mathrm{Kh}^{i,j}(T_{3,q}, \mathbb{Q}) = \dim_{\mathbb{Z}_p} \mathrm{Kh}^{i,j}(T_{3,q}, \mathbb{Z}_p).$$

- By the universal coefficient theorem, the above equality guarantees there is no odd torsion in $\mathrm{Kh}^{**}(T_{3,q})$.
- In 2017 Mounir computed $\mathrm{Kh}^{**}(T_{3,q}, \mathbb{Z}_2)$.

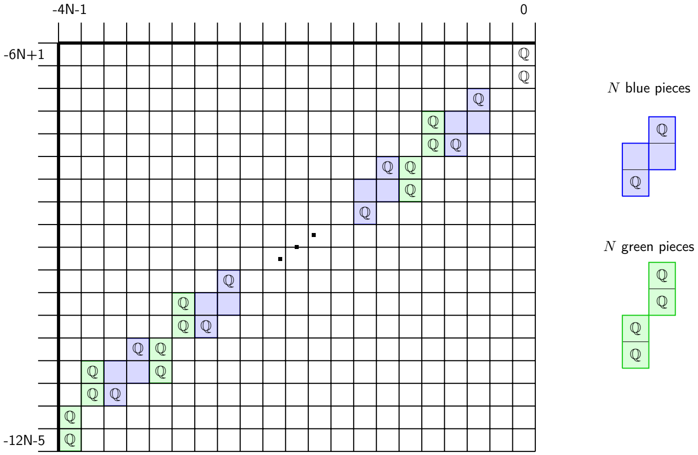


Figure: $\mathrm{Kh}(T_{3,3N+1}, \mathbb{Q})$ as computed by Turner, 2006 (Stošić, 2007)

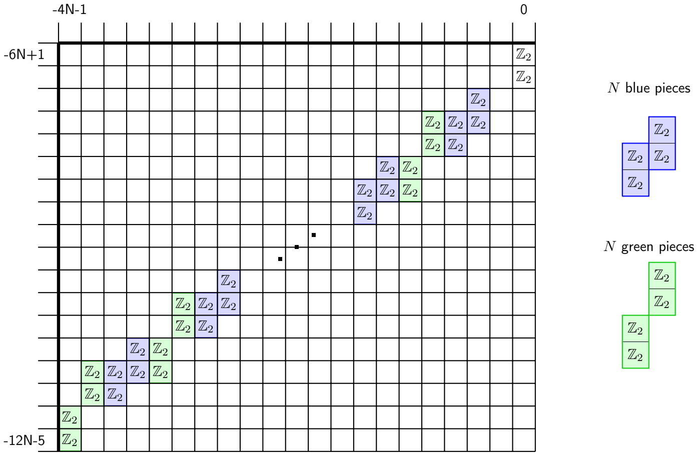


Figure: $\mathrm{Kh}(T_{3,3N+1}, \mathbb{Z}_2)$ as computed by Mounir, 2017

# The blue pieces and the green pieces

The universal coefficient theorem tells us

$$\mathrm{Kh}^{i,j}(L,\mathbb{Z}_2) \cong \big(\mathrm{Kh}^{i,j}(L)\otimes\mathbb{Z}_2\big)\oplus \mathrm{Tor}\big(\mathrm{Kh}^{i+1,j}(L),\mathbb{Z}_2\big)$$

$$\mathrm{Kh}^{i,j}(L,\mathbb{Q}) \cong \mathrm{Kh}^{i,j}(L)\otimes\mathbb{Q}$$

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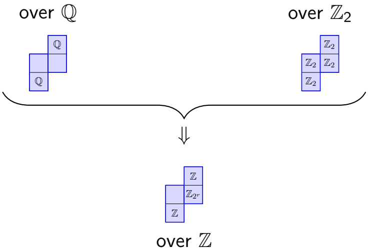


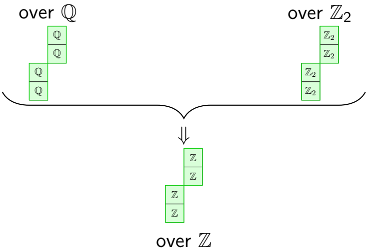

# Computing $\mathrm{Kh}^{**}(T_{3,q})$

Given Turner and Mounir's computations, the following theorem determines $\mathrm{Kh}^{**}(T_{3,q})$.

**Theorem (C., Lowrance, Sazdanovic, Summers)**

*$Kh^{**}(T_{3,q})$ has only $Z_2$-torsion.*

Proof: Repeat Shumakovitch's argument that $\mathbb{Z}_2$H-thin links have only $\mathbb{Z}_2$-torsion. Shown in this talk for $T_{3,3N+1}$ but similar for $T_{3,3N}$ and $T_{3,3N-1}$. □

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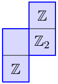
Therefore all blue pieces look like the following:




over $\mathbb{Z}$

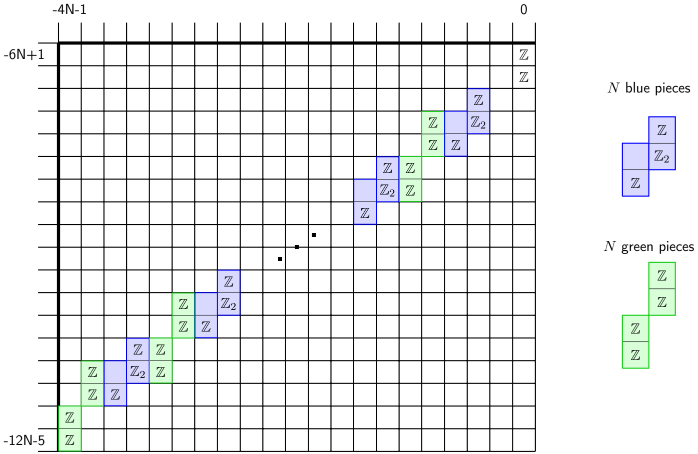
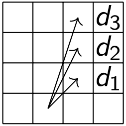


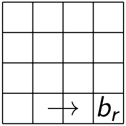

Figure: $\mathrm{Kh}^{**}(T_{3,3N+1})$ as determined by the previous theorem

# Spectral Sequences on $\mathrm{Kh}^{**}(L, \mathbb{Z}_2)$

To prove that $\mathrm{Kh}^{**}(T_{3,3N+1})$ has only $\mathbb{Z}_2$-torsion, we will make use of two spectral sequences with first page $\mathrm{Kh}^{**}(T_{3,3N+1}, \mathbb{Z}_2)$.

| | Turner | Bockstein |
|---|---|---|
| Pages and differentials | $(E_r, d_r)$ | $(B_r, b_r)$ |
| First page | $\mathrm{Kh}(T_{3,3N+1}, \mathbb{Z}_2)$ | $\mathrm{Kh}(T_{3,3N+1}, \mathbb{Z}_2)$ |
| Infinity page | $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)_0$ | $\mathrm{Free}(\mathrm{Kh}(T_{3,3N+1})) \otimes \mathbb{Z}_2$ |
| Degree on page $r$ | $(1, 2r)$ | $(1, 0)$ |



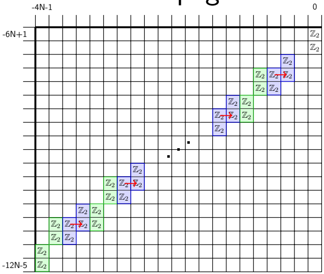



## Theorem

*If the $\mathbb{Z}_2$-Bockstein spectral sequence $(B_r, b_r)$ collapses on the $r^{th}$ page, then there is no $2^r$ torsion in homology.*

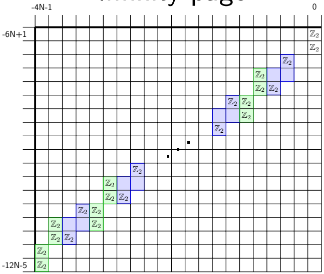
Goal: show that the Bockstein sequence on $Kh^{**}(T_{3,q}, \mathbb{Z}_2)$ collapses on the 2nd page. WTS red arrows are isomorphisms.

First page



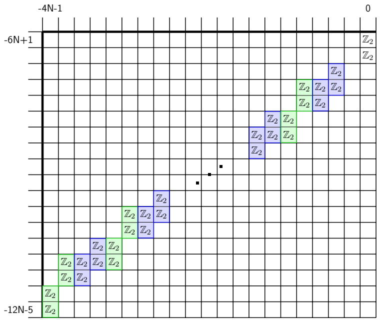

$\Rightarrow$

Infinity page



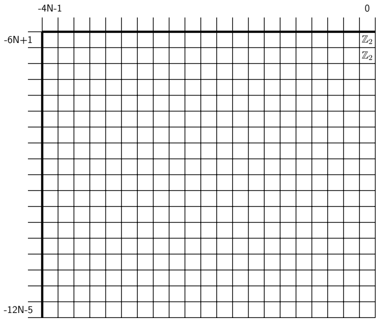
# Turner Spectral Sequence on $\mathrm{Kh}(T_{3,3N+1}, \mathbb{Z}_2)$

First page




$\Rightarrow$

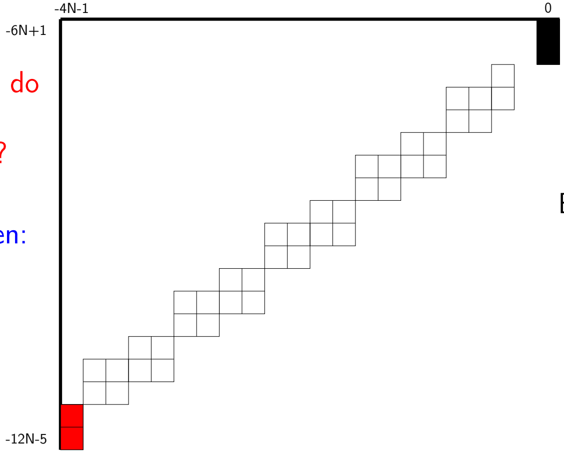
Infinity page



# How does everything cancel in Turner's spectral sequence?

Which squares do red squares cancel with?

Blue and green: already paired off




Black squares: survive to infinity page

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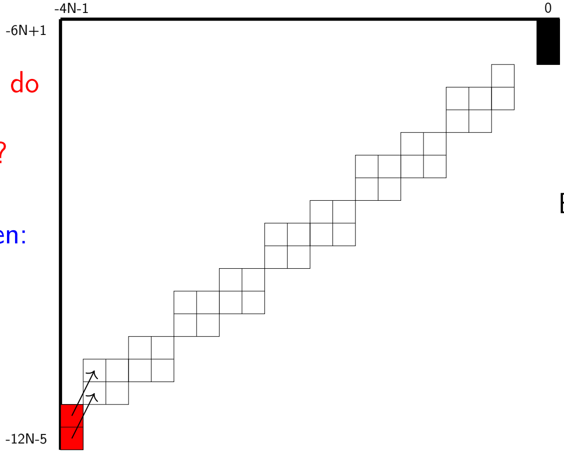


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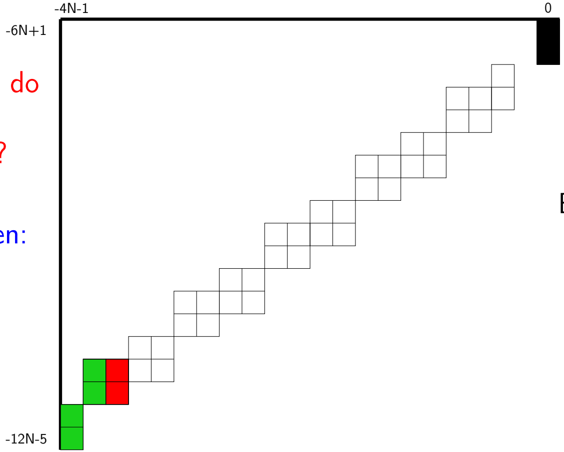


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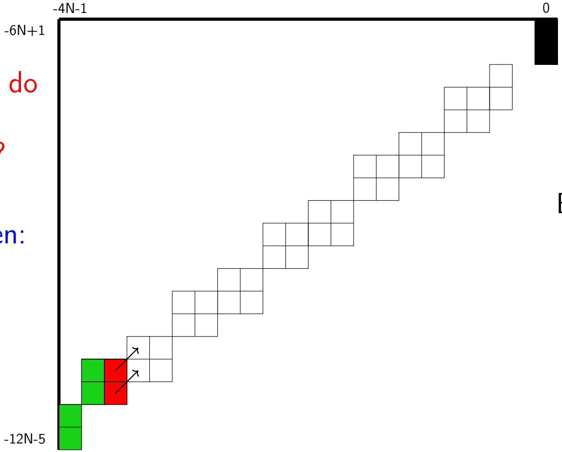


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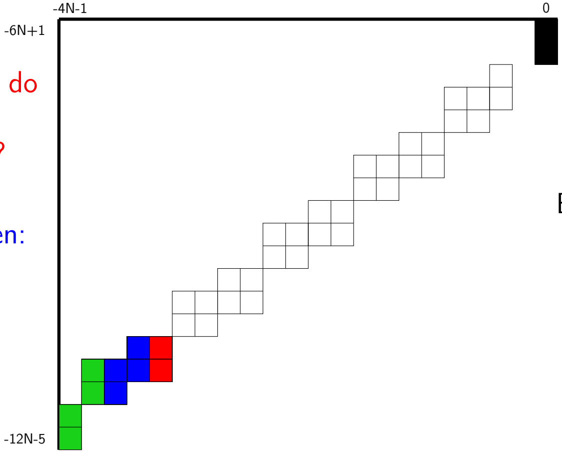


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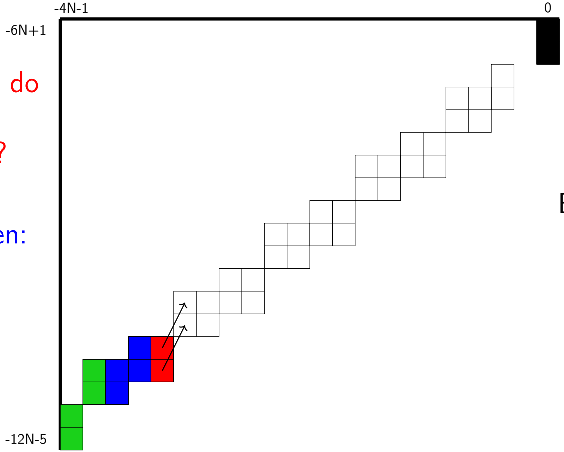


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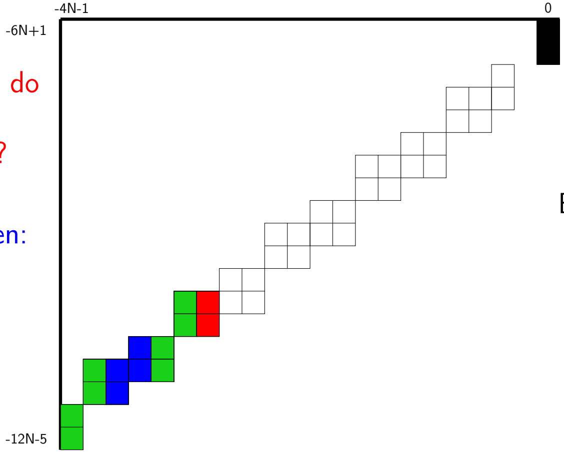


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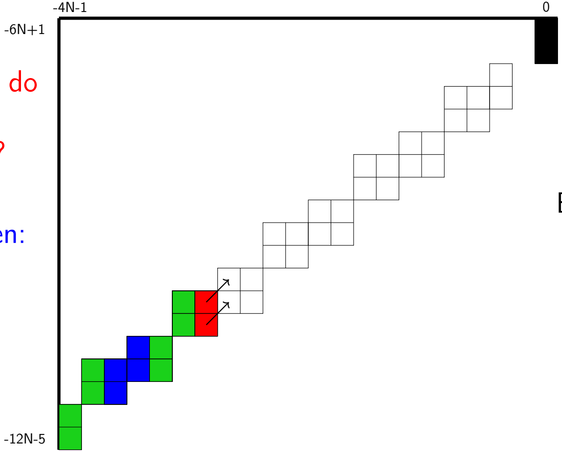


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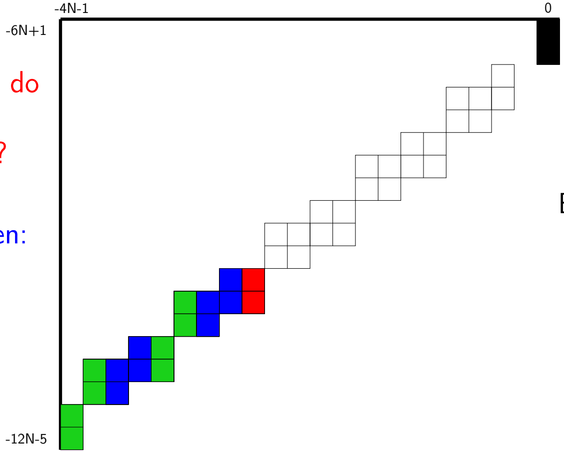


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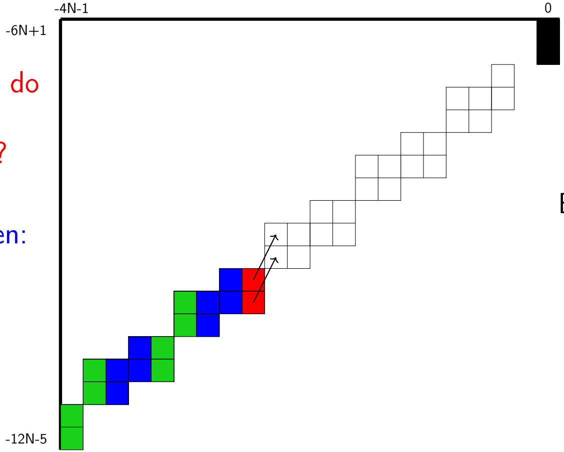


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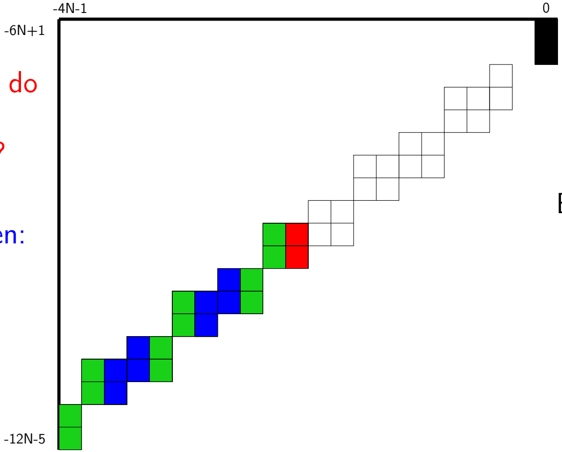


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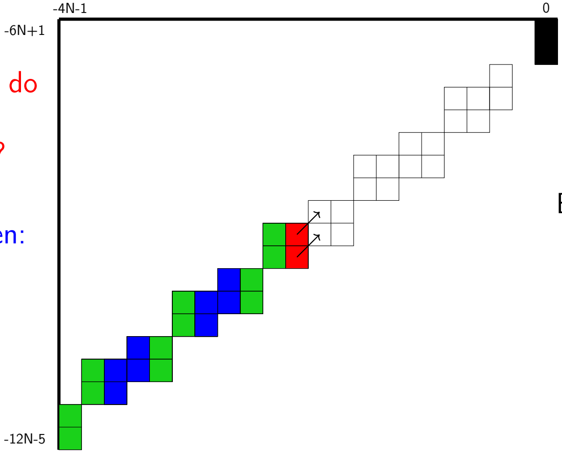


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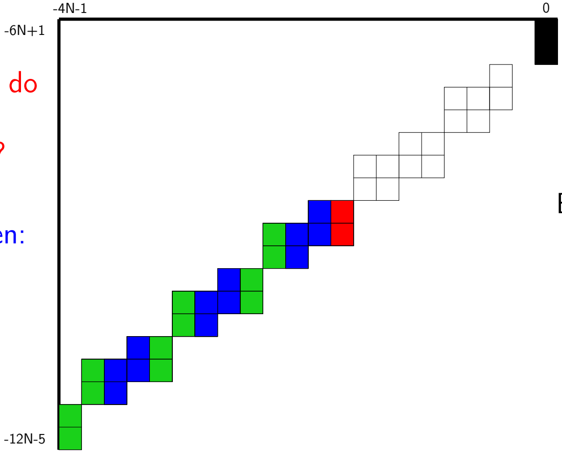


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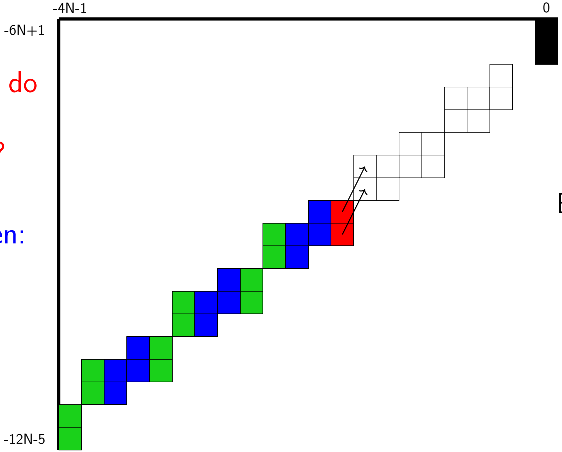


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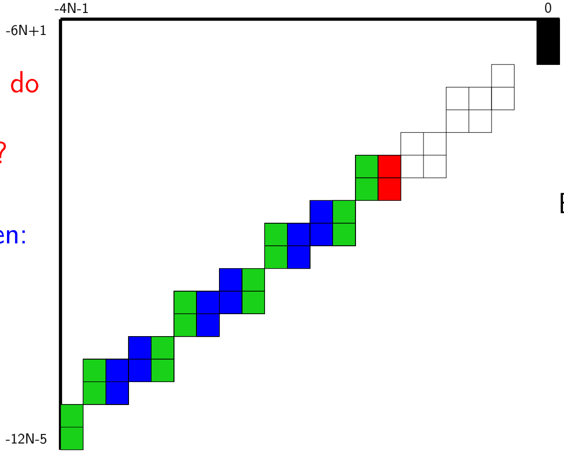


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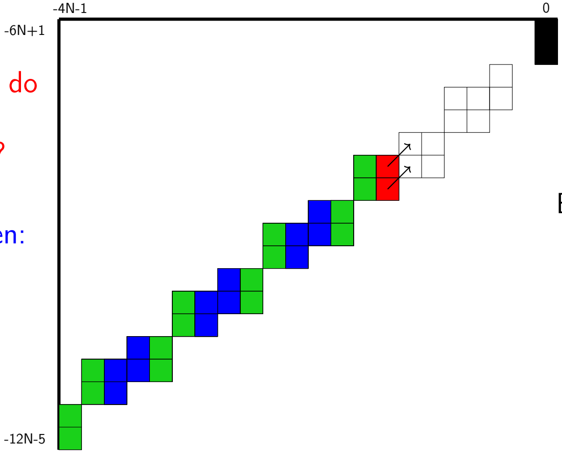


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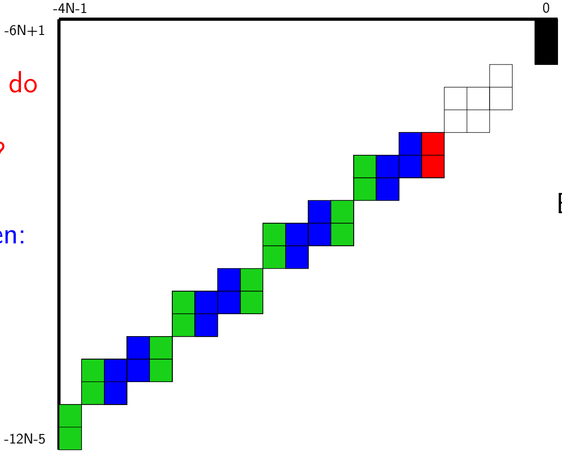


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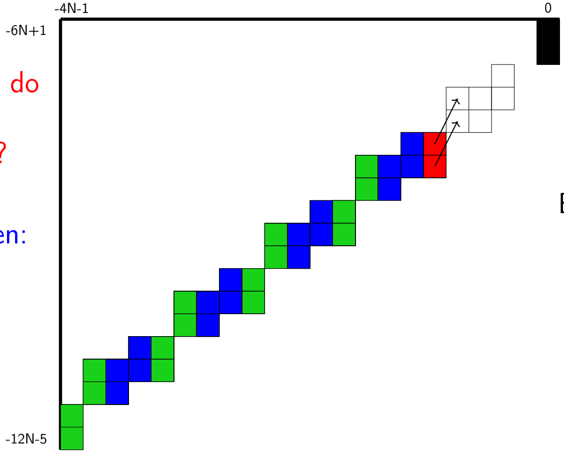


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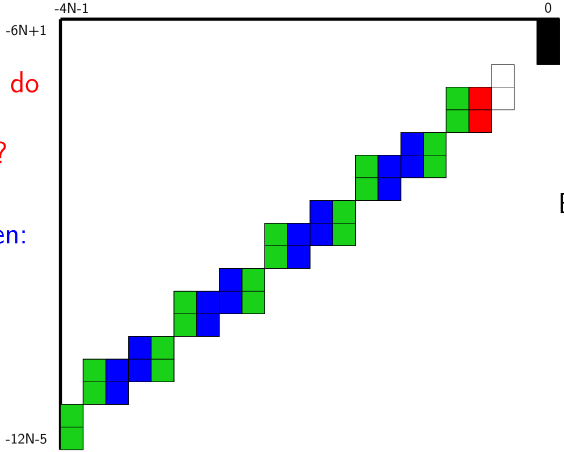


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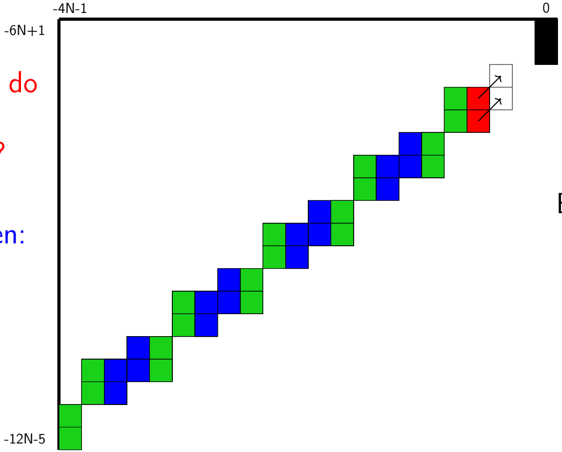


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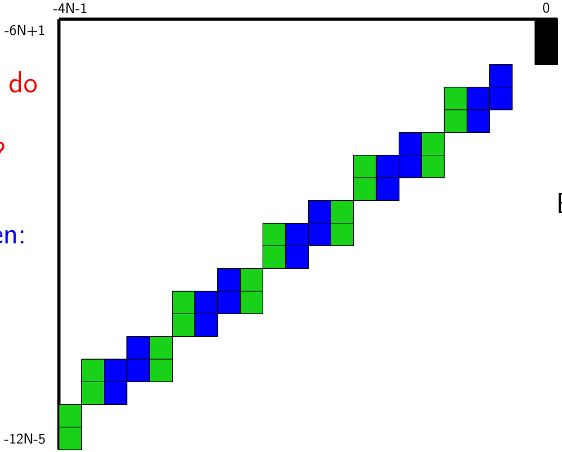


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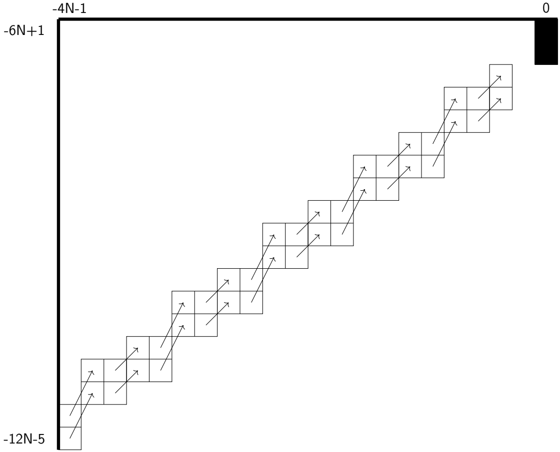
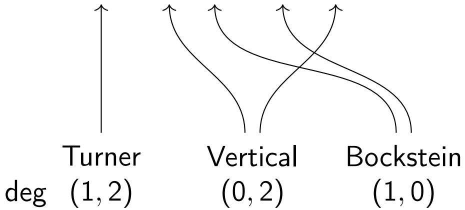
# Conclusion: The following maps are isomorphisms



# Vertical differentials

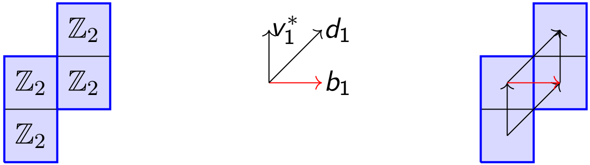
Shumakovitch showed there is a differential $v_1^*$ on $\mathrm{Kh}^{**}(L, \mathbb{Z}_2)$ of bidegree $(0, 2)$ such that

- $v_1^*$ is acyclic (Shumakovitch, 2004)
- $d_1 = v_1^* \circ b_1 + b_1 \circ v_1^*$ (Shumakovitch, 2009)



# Differentials and blue pieces

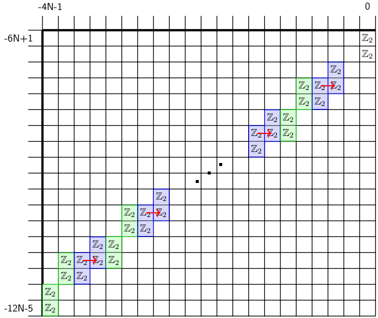
$$d_1 = v_1^* \circ b_1 + b_1 \circ v_1^*$$




- $b_1 : \mathbb{Z}_2 \to \mathbb{Z}_2$ is either 0 or an isomorphism
- In the diagram on the right, we know all black arrows are isomorphisms
- Therefore the equation above guarantees the red arrows are isomorphisms and thus cancel after the first page of $(B_r, b_r)$

# Bockstein Spectral Sequence on Kh($T_{3,3N+1}, \mathbb{Z}_2$)

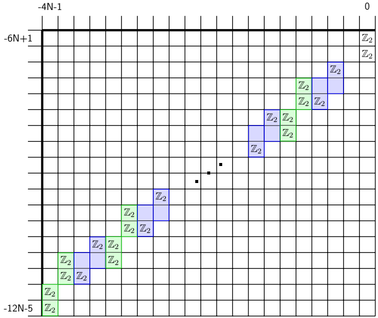
First page




$\mathrm{Kh}^{**}(T_{3,3N+1}, \mathbb{Z}_2)$

$\Rightarrow$

Second page = infinity page




$\mathrm{Free}(\mathrm{Kh}^{**}(T_{3,3N+1})) \otimes \mathbb{Z}_2$

# Future directions

## Theorem (From Murasugi)

*Every 3-braid is conjugate to one of the following:*

$$\Omega_0 = \{\Delta^{2n} \mid n \in \mathbb{Z}\} \qquad \Omega_3 = \{\Delta^{2n+1} \mid n \in \mathbb{Z}\}$$
$$\Omega_1 = \{\Delta^{2n}\sigma_1\sigma_2 \mid n \in \mathbb{Z}\} \qquad \Omega_4 = \{\Delta^{2n}\sigma_1^{-p} \mid n \in \mathbb{Z}\}$$
$$\Omega_2 = \{\Delta^{2n}(\sigma_1\sigma_2)^2 \mid n \in \mathbb{Z}\} \qquad \Omega_5 = \{\Delta^{2n}\sigma_2^{q} \mid n \in \mathbb{Z}\}$$
$$\Omega_6 = \{\Delta^{2n}\sigma_1^{-p_1}\sigma_2^{q_1}\dots\sigma_1^{-p_r}\sigma_2^{q_r} \mid n \in \mathbb{Z}\}$$

- $\Omega_0, \Omega_1, \Omega_2$ are torus links
- We have computed Kh of $\Omega_3$ over $\mathbb{Q}$ and $\mathbb{Z}_2$ and the same argument presented here goes through the same way
- $\Omega_4, \Omega_5, \Omega_6$ consist of an alternating braid word on top of a torus link. Use LES in Kh and what we know about the separate pieces to argue there is only $\mathbb{Z}_2$ torsion?

Thank you!