A Categorification of the Vandermonde Determinant

Alex Chandler

North Carolina State University

May 7, 2018

Categorification (a philosophy)

Categorification is the process of finding category theoretic analogues of set theoretic ideas by "adding extra structure":

categorification →

sets	categories
elements	objects
equations between elements	isomorphisms between objects
functions	functors

← decategorification

Decategorification is the reverse process (forgetting the extra structure)

Example: a categorification of $\mathbb{N}$

Objects: f.d. **k**-vector spaces

Morphisms: **k**-linear maps

k a field. The category <u>**k**-Vect</u> categorifies $\mathbb{N}$. Decategorify by taking dimension.

- V decategorifies to $\dim V$
- $V \oplus W$ decategorifies to $\dim V + \dim W$
- $V \otimes W$ decategorifies to $\dim V \ \dim W$

Categorifying $\mathbb{Z}$

Objects: bounded chain complexes

Morphisms: chain maps

The category $\mathcal{C}^b(\mathbf{k}\text{-Vect})$ categorifies $\mathbb{Z}$.

- C_* decategorifies to $\chi(C_*) = \sum_i (-1)^i \dim C_i$
- $C_* \oplus D_*$ decategorifies to $\chi(C_*) + \chi(D_*)$
- $C_* \otimes D_*$ decategorifies to $\chi(C_*)\chi(D_*)$

A classic example from topology

- Δ: a simplicial complex, $c_i = \#$ faces of dim i
- The Euler characteristic of Δ is

$$\chi(\Delta) = \sum_{i \geq 0} (-1)^i c_i$$

- $C_i(\Delta)$: free abelian group generated by faces of dimension i
- $d : C_k \to C_{k-1}$ sends

$$[v_{i_1}, \ldots, v_{i_k}] \mapsto \sum_j (-1)^j [v_{i_1}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_k}]$$

- $\chi(C_*(\Delta)) = \chi(\Delta)$

The plan for this talk

The **Vandermonde determinant** is defined as

$$V_n = \begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{\pi(1)} x_2^{\pi(2)} \ldots x_n^{\pi(n)}$$

- Categorify (evaluations of) V_n for $x_1, \ldots, x_n \in \mathbb{N}$
- Accomplish this in a way analogous to Khovanov's categorification of the Kauffman bracket

Khovanov homology

Categorifies the Kauffman bracket

$$\langle K \rangle = \sum_{\alpha \in \{0,1\}^n} (-1)^{h(\alpha)} q^{h(\alpha)} (q + q^{-1})^{s(\alpha)}$$

where K is a link diagram with n crossings,

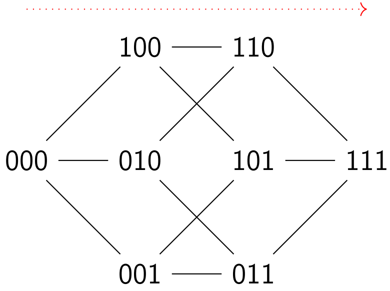
$$h(\alpha) = \#1's \text{ in } \alpha,$$

$$s(\alpha) = \#\text{circles in } \alpha\text{-smoothing}$$

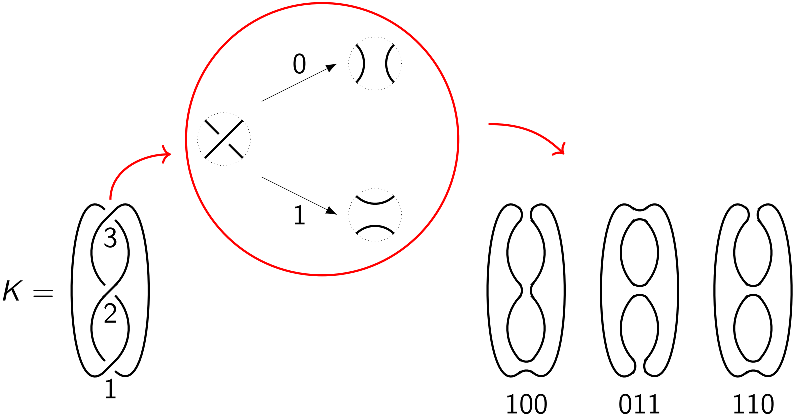
The n-cube $C_n = \{0,1\}^n$

- Partially ordered set with vertices $\{0,1\}^n$
- Cover relation (edge) when you change a 0 to a 1:

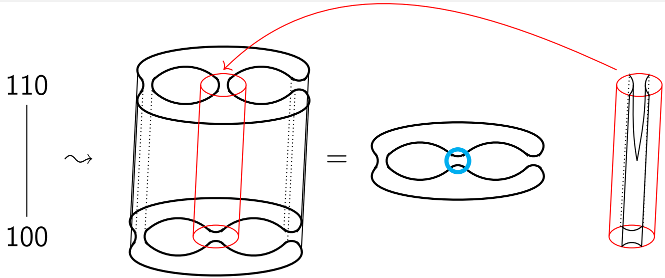
C_3 :



n-tuples of 1's and 0's encode **smoothings** of *K*



Edges encode **cobordisms** between smoothings



- Start with (100-smoothing) $\times [0,1]$
- Remove cylindrical neighborhood of changing crossing
- Replace with a saddle

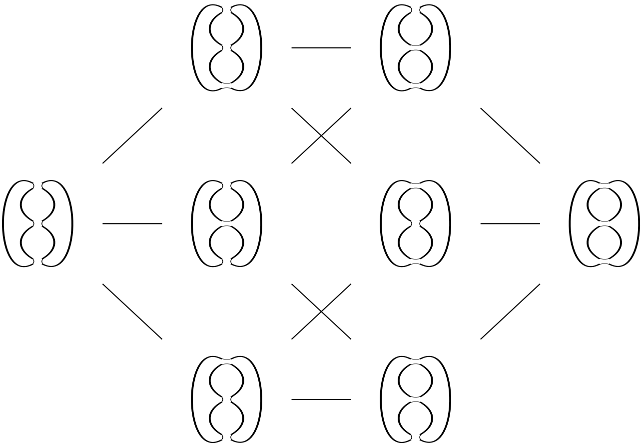
Category of cobordisms

Objects: closed 1-dim manifolds

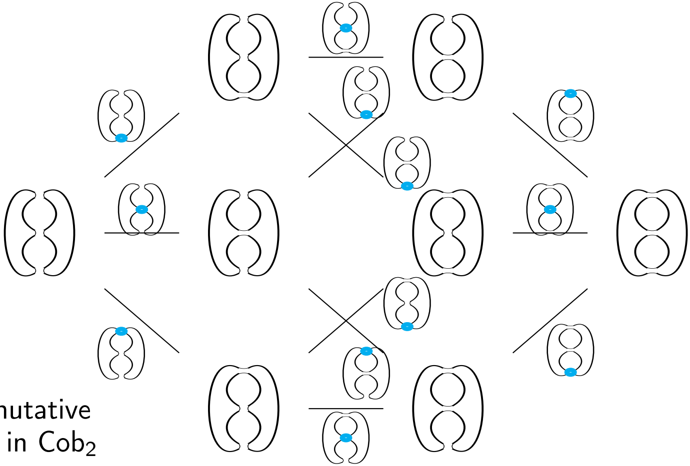
Morphisms: 2-dim cobordisms

The category Cob_2 contains smoothings and cobordisms as its objects and morphisms

Replace vertices in C_n by corresponding smoothings



Replace edges in C_n by cobordisms

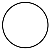


A commutative diagram in Cob_2

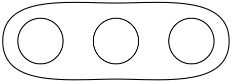
2D TQFTs and Frobenius algebras

A 2D TQFT is a monoidal functor from Cob_2 to $\mathbf{k}$-Vect.

- assigns a $\mathbf{k}$-vector space A to each circle

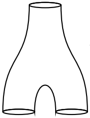


$\longrightarrow \quad A$



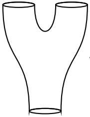
$\longrightarrow \quad A \otimes A \otimes A \otimes A$

- assigns linear maps to cobordisms



$\longrightarrow \left(m : A \otimes A \to A \right)$ multiplication

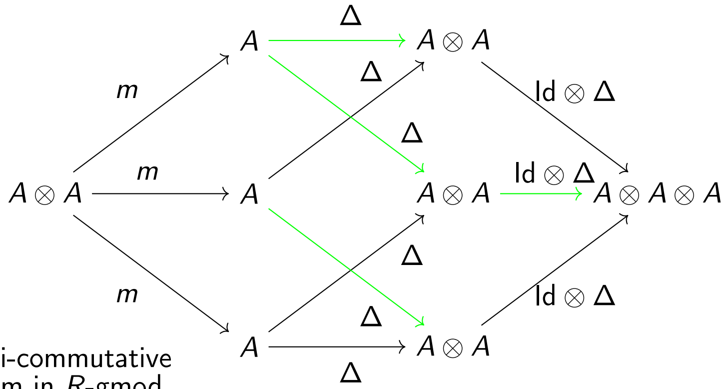
$\longrightarrow \left(\eta : R \to A \right)$ unit



$\longrightarrow \left(\Delta : A \to A \otimes A \right)$ comultiplication

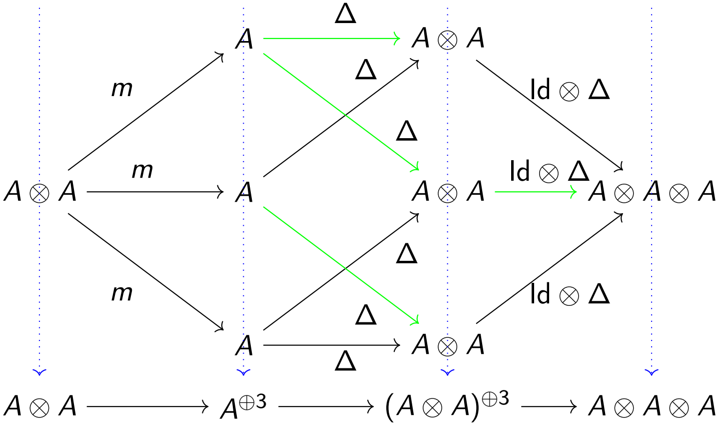
$\longrightarrow \left(\epsilon : A \to R \right)$ counit

Apply a 2D TQFT (with $q\dim A = q + q^{-1}$)



An anti-commutative diagram in R-gmod

Direct sum down ranks and get a chain complex



Theorem (Khovanov)

The (shifted) homology groups of this chain complex are link invariants and the graded Euler characteristic of this complex is equal to the Kauffman bracket

$$\sum_{i\in\mathbb{Z}}(-1)^i q\dim H^i = \langle K\rangle$$

- The (shifted) Khovanov homology groups give a strictly stronger link invariant than the Jones polynomial
- Khovanov homology is a functor. That is, cobordisms between links induce maps between Khovanov homology groups

Why did this construction work?

- The Kauffman bracket is a rank alternating sum over a ranked poset $P = C_n$

$$\sum_{x \in P} (-1)^{r(x)} f(x)$$

- Every interval of length 2 in C_n is a diamond (i.e. C_n is **thin**)
- There is a $\{+1, -1\}$ edge coloring of C_n for which each diamond has an odd number of -1's (a **balanced coloring**)

Categorifying Vandermonde

$$V_n = \sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_1^{\pi(1)} x_2^{\pi(2)} \dots x_n^{\pi(n)}$$

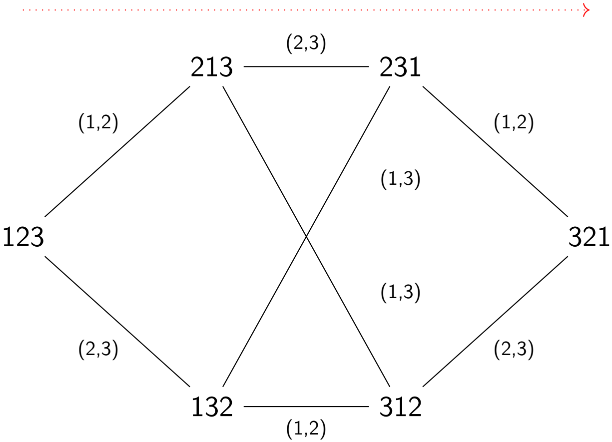
- S_n has a thin poset structure: the **Bruhat order**
- $\mathrm{inv}(\pi)$ is the rank function for this ordering
- The Bruhat order has a balanced coloring

We're in business!

Bruhat Order on S_n

- An **inversion** in a permutation π is a pair (i, j) with $i < j$ and $\pi(i) > \pi(j)$
- $\mathrm{inv}(\pi)$ denotes the number of inversions of π
- Bruhat order on S_n has a vertex for each $\pi \in S_n$
- Has an edge (cover relation) $\pi \lessdot \sigma$ whenever σ is gotten from π (in one line notation) by transposing a non-inversion pair for which $\mathrm{inv}(\sigma) = \mathrm{inv}(\pi) + 1$

E.g. Bruhat Order on S_3



Colored Cobordisms: Cob_2^n

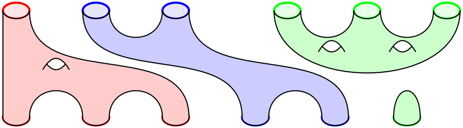
Objects: $[n]$-colored closed 1-manifolds

Morphisms: color preserving cobordisms

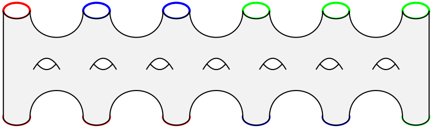
Let $[n] = \{1, 2, ..., n\}$. The category Cob_2^n has

- Objects: closed 1-manifolds with each connected component given a color from $[n]$
- Morphisms: 2-dimensional manifolds for which each connected component has monochromatic boundary

For example, let $M =$  and $N =$ 



is a colored cobordism from M to N, but not



Permutations (vertices) encode colored smoothings

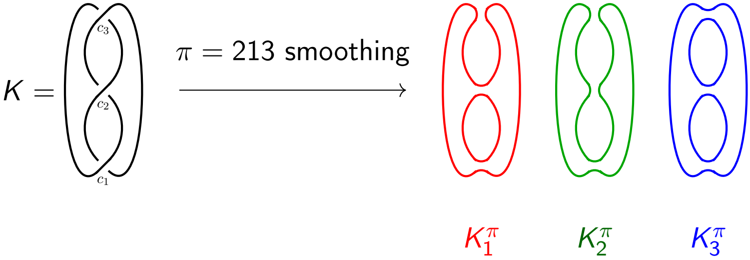
K a link diagram with crossings $c_1, \ldots, c_n$ and $\pi \in S_n$

- The π-**smoothing** of K is

$$K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \text{Ob Cob}_2^n$$

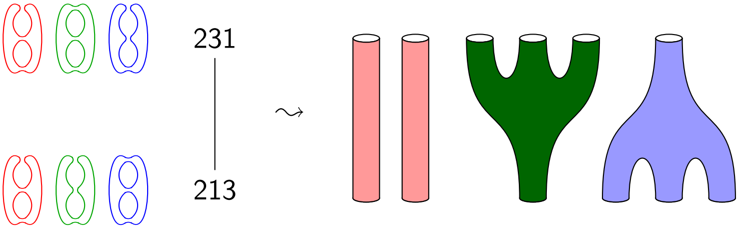
- K_i^π is gotten from K by giving $c_1, c_2, \ldots, c_{\pi(i)}$ 1-smoothings and all other crossings 0-smoothings
- All components of K_i^π are colored i

π-Smoothing Example

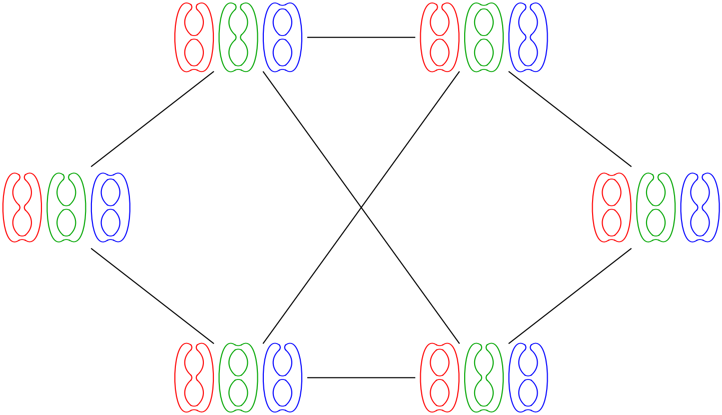


Edges encode colored cobordisms

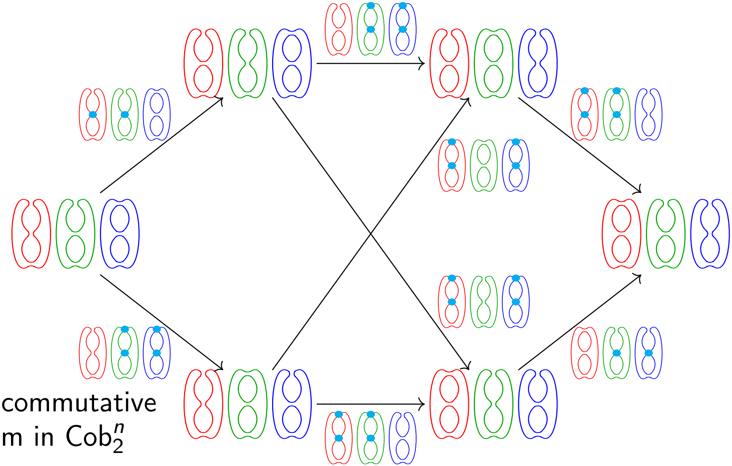
If $\pi \lessdot \sigma$ then $K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \text{Ob Cob}_2^n$ and
$K^\sigma = K_1^\sigma \amalg K_2^\sigma \amalg \ldots \amalg K_n^\sigma \in \text{Ob Cob}_2^n$
differ at exactly two colors. Use connected genus 0 cobordisms on the colored pieces which differ, and identity (cylinders) on pieces which do not change



Replace vertex π with π-smoothing



Replace edges with colored cobordisms

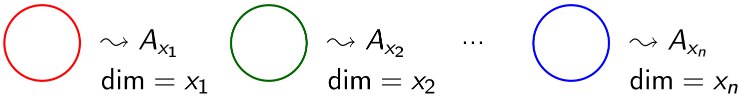


A (non) commutative diagram in Cob_2^n

2D colored TQFTs

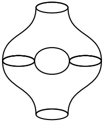
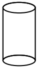
Definition

- A **colored TQFT** is a monoidal functor $F : \mathrm{Cob}_2^n \to \mathbf{k}\text{-Vect}$ which restricts to a TQFT on each color.



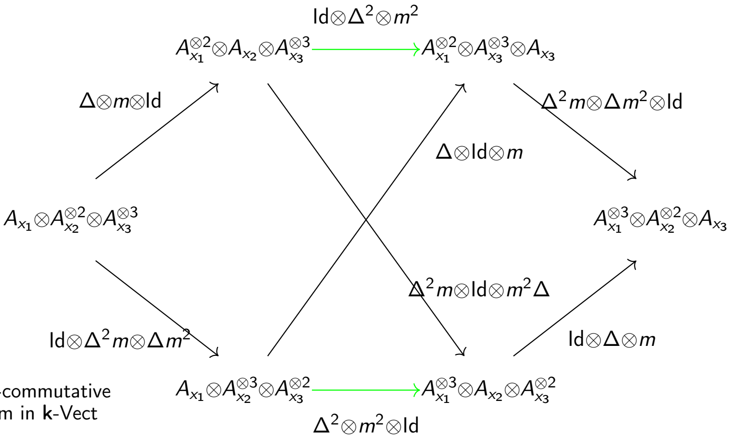
Special TQFTs and special Frobenius algebras

A 2D TQFT F is **special** if the following condition holds:

$F\Big($  $\Big) = F\Big($  $\Big) \iff \mu \circ \Delta = 1$

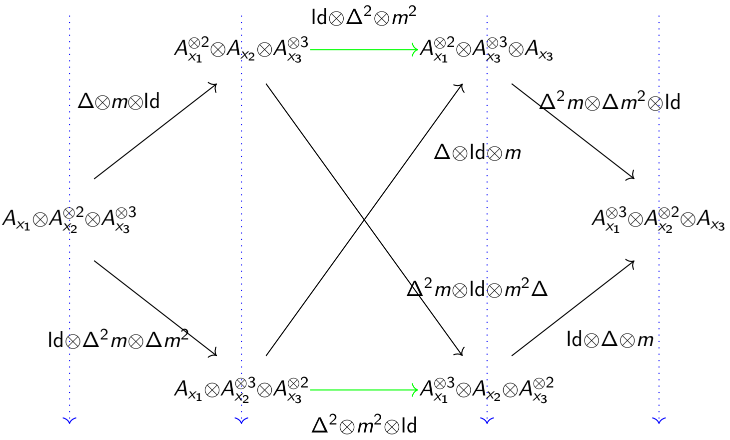
A 2D colored TQFT is special if its restriction to each color is a special TQFT

Apply a Special Colored TQFT



An anti-commutative diagram in **k**-Vect

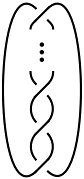
Direct sum down ranks to get a chain complex



Theorem (C.)

Let K be the alternating two strand braid diagram of the $(2, n)$-torus knot. Then the Euler characteristic of this chain complex is equal to the Vandermonde determinant

$$V_n = \sum_{i \geq 0} (-1)^i dim H^i$$

$K =$ 

What's Next?

Questions:

- Is this categorification functorial?
- What kinds of polynomials do we recover for arbitrary knots?
- Do specific classes of knots correspond to known classes of polynomials?
- Relation to $V_n = x_1 \dots x_n \prod_{i<j}(x_j - x_i)$

Thank you!