Out and about with:
Topological Quantum Field Theories

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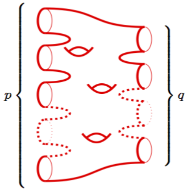
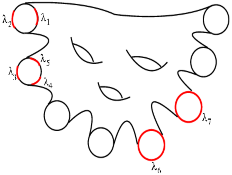
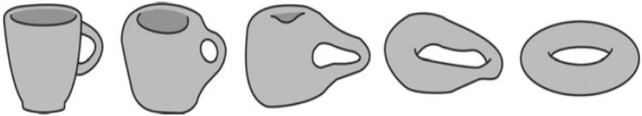


May 7, 2018



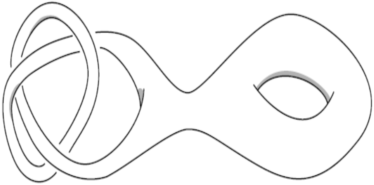
What is Topology?

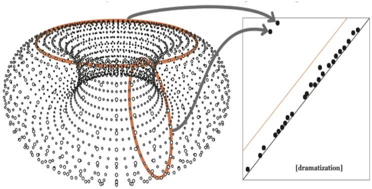
In topology we study properties of geometric objects which are unaffected by continuous deformations.



Why care about Topology?

Topology is fun!



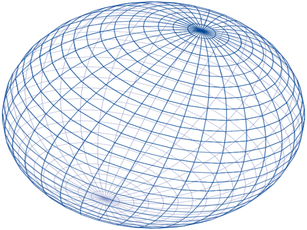


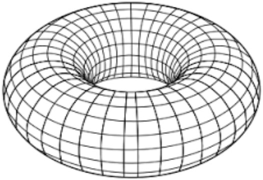
Topology helps us solve interesting problems

- Big data analysis
- Inscribed rectangle problem
- Fundamental Theorem of Algebra
- 2016 Nobel prizes in Physics and Chemistry

What do Topologists do?

Basic Question: How can we tell when two spaces are the same or different?

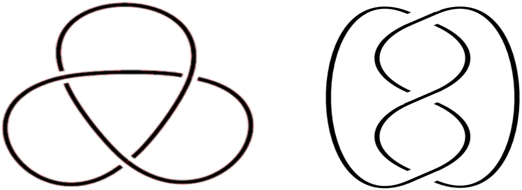




Can we continuously deform the sphere into the torus?

Basic Questions in Topology

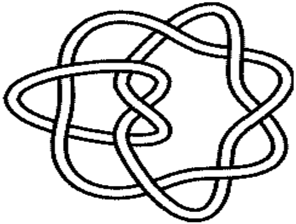
Basic Question: How can we tell when two spaces are the same or different?

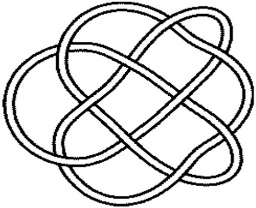


These are the same (not hard to deform one into the other... try it!)

Basic Questions in Topology

Are these knots the same?



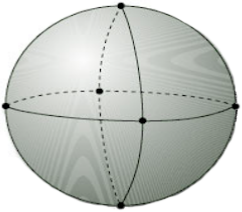


Not easy to see by hand. We need a more systematic way to answer this!

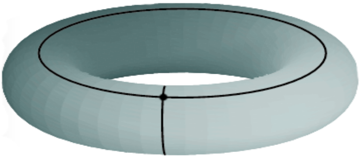
Topological Invariants

A topological invariant is any mathematical object which we can associate to a space which does not change when we continuously deform the space.

E.g. Euler Characteristic: $\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces}$

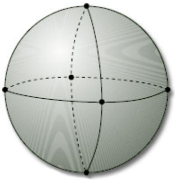


$\chi = 6 - 12 + 8 = 2$



$\chi = 1 - 2 + 1 = 0$

When Euler Doesn't Do the Trick



$\chi = 2$ $\chi = 2$

χ can't tell the difference!!

Categorification: Making a Good Thing Even Better

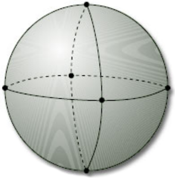
Categorification is the process of upgrading a mathematical object to a more powerful one (which contains the same info... and some extra)

E.g. The euler characteristic $\chi(X)$ of a space X is categorified by the singular homology $H_*(X) = \bigoplus_{i\geq 0} H_i(X)$ of the space.

$$\chi(X) = \sum_{i=1}^{\infty} (-1)^i \text{rank} H_i(X)$$

Euler Fails but Homology Prevails!

χ can't tell the difference but H_* can!

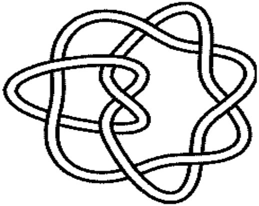


$$\chi = 2$$
$$H_* = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)}$$

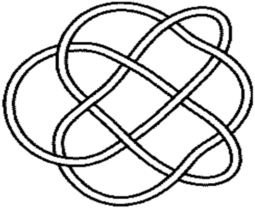
$$\chi = 2$$
$$H_* = (\mathbb{Z} \oplus \mathbb{Z})_{(0)}$$

Distinguishing Knots

The Jones polynomial is good at telling knots apart:



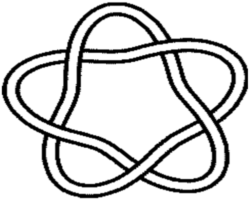
$J(q) = -q^4 + q^3 - q^2$
$+2q - 1 + 2q^{-1} - q^{-2}$
$+q^{-3} - q^{-4}$



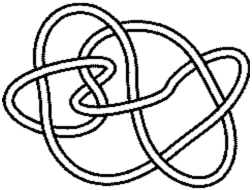
$J(q) = q^3 - 4q^2 + 8q$
$-11 + 15q^{-1} + 15q^{-3}$
$-12q^{-4} + 8q^{-5} - 4q^{-6} + q^{-7}$

Polynomials are different $\implies$ knots are different

When Jones Doesn't Do the Trick



$$J(q) = 2q^{-1} - 3q^{-2} + 5q^{-3} - 5q^{-4} + 5q^{-5} - 5q^{-6} + 3q^{-7} - 2q^{-8} + q^{-9}$$

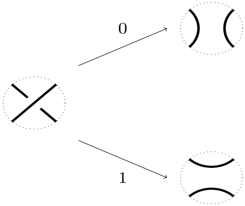


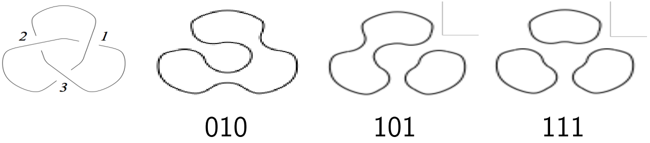
$$J(q) = 2q^{-1} - 3q^{-2} + 5q^{-3} - 5q^{-4} + 5q^{-5} - 5q^{-6} + 3q^{-7} - 2q^{-8} + q^{-9}$$

Polynomials same $\not\Rightarrow$ knots are the same

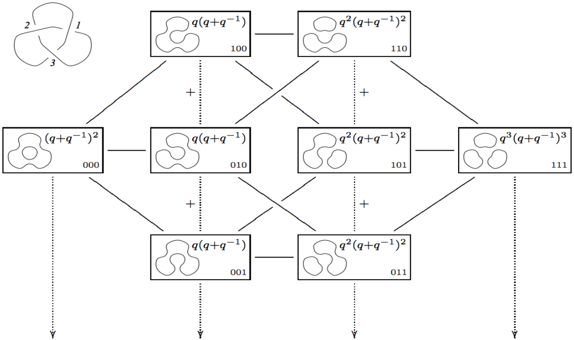
Let's categorify the Jones Polynomial!

How to define Jones? First we need *smoothings*





Definition of Jones Polynomial

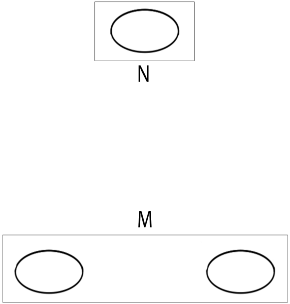


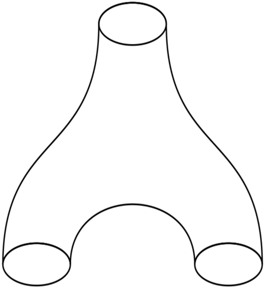
$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

To categorify, apply a 2-dimensional Topological Quantum Field Theory to this picture!

What are TQFTs?

If M and N are smoothings, a *cobordism* from M to N is a surface which connects M to N.

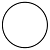
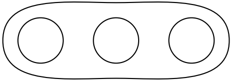




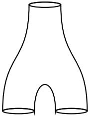
pants cobordism from
2 circles to 1 circle

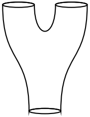
What are TQFTs?

A 2-dimensional TQFT assigns a vector space V to each circle

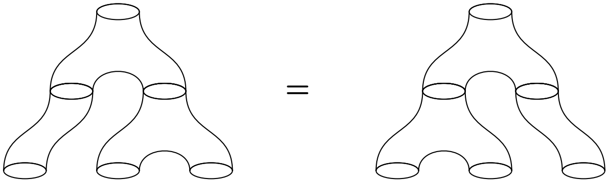
 $\longrightarrow$ V  $\longrightarrow$ $V \otimes V \otimes V \otimes V$

and assigns linear maps to cobordisms

 $\longrightarrow \left(m : V \otimes V \to V \right)$ multiplication

 $\longrightarrow \left(\Delta : V \to V \otimes V \right)$ comultiplication

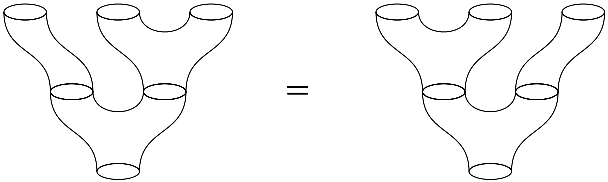
Stretchy Pants and Frobenius Algebras



$$\Longrightarrow \qquad m \circ (Id_A \otimes m) = m \circ (m \otimes Id_A)$$

That is, m is an associative multiplication on A

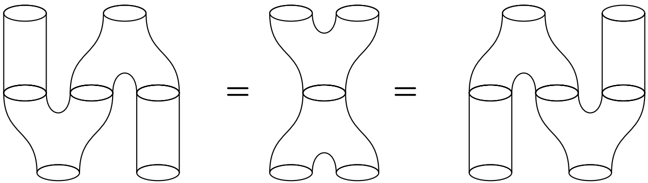
Stretchy Pants and Frobenius Algebras



$$\implies \qquad (Id_A \otimes \Delta) \circ \Delta = (\Delta \otimes Id_A) \circ \Delta$$

That is, Δ is a coassociative comultiplication on A

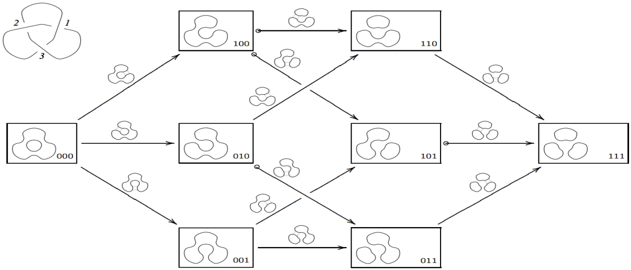
TQFTs as Frobenius Algebras



These relations together tell us that V is a *Frobenius Algebra*.

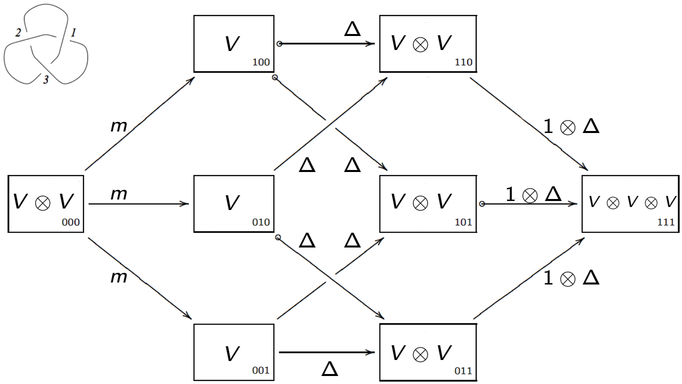
$$\{\text{2-dim TQFTS}\} \longleftrightarrow \{\text{Frobenius Algebras}\}$$

Categorifying Jones: Khovanov Homology



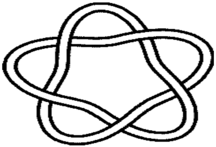
Adorn edges of the diagram with cobordisms between the smoothings and apply a TQFT!

Categorifying Jones: Khovanov Homology

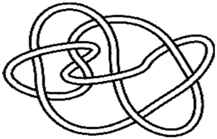


The homology of this complex is called the Khovanov homology

Jones Fails but Khovanov Prevails!



j \ r	-5	-4	-3	-2	-1	0	χ
-3						1	1
-5						1	1
-7				1			1
-9							0
-11		1	1				0
-13							0
-15	1						-1



j \ r	-8	-7	-6	-5	-4	-3	-2	-1	0	χ
-1									2	2
-3								2	1	-1
-5							3	1		2
-7						2	2			0
-9					3	3				0
-11				2	2					0
-13			1	3						-2
-15		1	2							1
-17		1								-1
-19	1									1


"That's all Folks!"


NCSU