Derived Categories and Mirror Symmetry in String Theory

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Abstract

We begin by introducing derived categories and describing why they are useful in a general setting. The rest of the paper will be dedicated to understanding an equivalence of the A-model and B-model of Dirichlet branes in string theory known as homological mirror symmetry. The B-model is realized mathematically as coherent sheaves on a Kähler manifold, and the A-model is realized as Lagrangian submanifolds of a Kähler manifold. Roughly, the statement of mirror symmetry in string theory is that the bounded derived category $D^b\mathrm{Coh}(M)$ of coherent sheaves on M is equivalent to the bounded derived category $D^b\mathrm{Fuk}(M^\vee)$ of Lagrangian submanifolds of the mirror manifold $M^\vee$ (this is called the Fukaya category).

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1 Introduction

In string theory, **branes** are smooth submanifolds of spacetime which can be thought of as a sort of generalization of point particles. p-branes (branes of dimension p) propagate through spacetime, sweeping out a $(p+1)$ dimensional volume. 0-dimensional branes can be thought of as point particles. 1-dimensional branes are what we call **strings**. 2-dimensional branes are called **membranes**, which is where the term brane originates. Strings, being 1-manifolds, can be either closed loops (these are called closed strings) or can have endpoints (these are called open strings). Open strings are allowed to propagate through spacetime subject to boundary conditions called **Dirichlet conditions**. The set of allowable endpoints of a string is called a **Dirichlet brane** or **D-brane**. In topological string theory, there are two mathematical models of D-branes: the A-model and the B-model. A-model D-branes are described mathematically as certain Lagrangian submanifolds of a Kähler manifold, and B-model D-branes are described as coherent sheaves on a Kähler manifold. We aim to understand a certain equivalence between these models, which (roughly) comes in the form of an equivalence of the derived category of coherent sheaves on a Kähler manifold M with the derived Fukaya category on the mirror manifold $M^\vee$.

In section 2 we will study derived categories. Section 3 will be an introduction to sheaf theory concluding with a definition of coherent sheaves. At this point of the paper we will understand the B-model of Dirichlet branes. Section 4 will give an introduction to manifold theory and symplectic structures, and section 5 will describe the Lagrangian intersection Floer chain complex for pairs of Lagrangian submanifolds of a symplectic manifold. Section 6 is where we find the definition of the Fukaya category, the final piece to the puzzle. Now we understand both the A-model and B-model of Dirichlet branes, and can make some sense of the mirror symmetry conjecture. Then in section 7, Kontsevitch's conjecture of mirror symmetry is stated, and some closing remarks are given.

2 Derived Categories

Derived categories were introduced by Verdier in his PhD thesis, and since then there has been a rapid development of the topic and its relatives. Derived categories appear in many fields of algebra, topology, algebraic geometry, and even mathematical physics. The main goal of this section is to give enough of an introduction to derived categories to later be able to understand Kontsevitch's conjecture of homological mirror symmetry: an equivalence between certain derived categories.

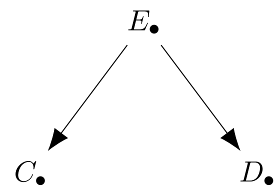
Definition 1. *Let C be a category, and W some class of morphisms in C. The* ***localization*** *$C[W^{-1}]$ is the category uniquely determined by the universal property:*

> *there exists a natural functor $C \to C[W^{-1}]$ and given any category D, a functor $F : C \to D$ factors uniquely over $C[W^{-1}]$ if and only if F sends all arrows in W to isomorphisms.*

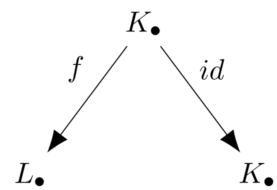
Morally, to construct the localization $C[W^{-1}]$, simply take the objects of C as objects of $C[W^{-1}]$ and for each morphism $f \in \mathrm{Hom}(O_1, O_2) \cap W$ add in a formal inverse $f^{-1} \in \mathrm{Hom}(O_2, O_1)$ and make the identifications $ff^{-1} = 1_{O_2}$ and $f^{-1}f = 1_{O_1}$.

In what follows, we would like to allow ourselves to work in a category in which chain complexes and homologies of chain complexes make sense (that is, we need the notions of a zero object, kernels, images, and cokernels). Without going into too much technical detail, suffice to say that **abelian categories** are exactly the categories where these notions make sense. For formal definitions, the reader is referred to [4] Chapter 2 Section 5.

Definition 2. *A morphism $f : K_\bullet \to L_\bullet$ of complexes in an abelian category is called a* ***quasi-isomorphism*** *if the induced map $H_n(K_\bullet) \to H_n(L_\bullet)$ is an isomorphism for all n. Chain complexes $C_\bullet$ and $D_\bullet$ are* ***quasi-isomorphic*** *if there exists a complex $E_\bullet$ and quasi-isomorphisms:*



Remark: If $f : K_\bullet \to L_\bullet$ is a quasi-isomorphism, then $K_\bullet$ and $L_\bullet$ are certainly quasi-isomorphic via:



but $K_\bullet$ and $L_\bullet$ need not have a quasi-isomorphism between them in order to be quasi-isomorphic.

For example, in the case of simplicial chain complexes of a triangulable topological space X, one can construct situations where we have two triangulations X_1 and X_2 with chain complexes $\mathcal{C}_\bullet(X_1)$ and $\mathcal{C}_\bullet(X_2)$ such that there is no chain map $f : \mathcal{C}_\bullet(X_1) \to \mathcal{C}_\bullet(X_2)$ which induces isomorphisms on homology groups. However, we know that the homology groups are indeed isomorphic, and would like to regard $\mathcal{C}_\bullet(X_1)$ and $\mathcal{C}_\bullet(X_2)$ as being quasi-isomorphic (after all, they are chain complexes for the same space!). We introduce $E_\bullet$ to extend the notion of quasi-isomorphism to this situation. One can often find a common refinement of the triangulations (playing the role of $E_\bullet$), making the simplicial chain complexes quasi-isomorphic.

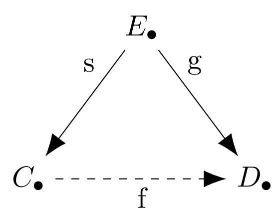
Definition 3. *The **bounded derived category** $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization of (the full subcategory of) bounded complexes in $\mathcal{A}$ at the class of quasi-isomorphisms. That is:*

- *Objects are bounded complexes in $\mathcal{A}$ (complexes with only finitely many nonzero terms)*
- *Each chain map $f : C_\bullet \to D_\bullet$ is a morphism in $D^b(\mathcal{A})$*
- *If $g : C_\bullet \to D_\bullet$ is a quasi-isomorphism, we also include a formal inverse $g^{-1} : D_\bullet \to C_\bullet$*
- *Include formal compositions $f_1 \circ f_2$ for all possible combinations whose terminal object and initial object coincide*
- *When f_1 and f_2 are true chain maps (neither arise from the 2nd or 3rd bullet), identify the composition $f_1 \circ f_2$ with their actual composition as chain maps.*
- *If g is a quasi-isomorphism, identify $g \circ g^{-1}$ and $g^{-1} \circ g$ with the corresponding identity chain maps.*

In the same way we can also define the bounded above and bounded below derived categories $D^-(\mathcal{A})$ and $D^+(\mathcal{A})$. Use $D(\mathcal{A})$ to denote a derived category without specifying which of the above we are talking about.

Facts/Remarks about the definition:

1. Since homotopy equivalences are quasi-isomorphisms, chain homotopic maps in the derived category are automatically identified.
2. Any morphism in $D(\mathcal{A})$ can be expressed as a **roof**:



   where s is a quasi-isomorphism, and we set $f = gs^{-1}$.

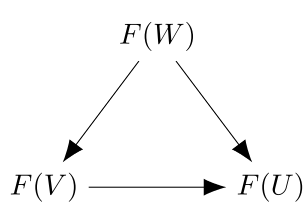
3. Derived categories allow one to tie together all 'derived functors' into one, called the 'total derived functor'. In a sense, derived categories are the proper setting to study derived functors (see [4] or [6] for details)
4. Since quasi-isomorphisms are now identified, we can freely replace an object $\mathcal{O} \in \mathcal{A}$ by any free, injective, or projective resolution, whichever is convenient at the time (again, see [4] or [6] for details).

Our goal is to provide enough background and motivation to at least understand Kontsevitch's conjecture which is now known as homological mirror symmetry (see section 7).

3 Coherent Sheaves

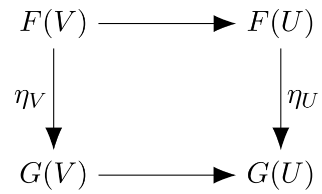
Let $\mathcal{A}$ be an abelian category, let X be a topological space, and let $\mathcal{U}$ denote the collection of open sets of X. A **Presheaf** on X with values in $\mathcal{A}$ is a rule F which does the following:

- For each open set U, $F(U)$ is an object in $\mathcal{A}$, and $F(\varnothing) = \{0\}$.
- For each inclusion $U \subseteq V$, F assigns a morphism $F(V) \to F(U)$.
- For each $U \subseteq V \subseteq W$, the following diagram commutes



Given presheafs F, G on X with values in $\mathcal{A}$, a **morphism of presheaves** is a rule η which does the following:

- For each open set U, η_U is a morphism $F(U) \to G(U)$ in $\mathcal{A}$.
- For each inclusion $U \subseteq V$, the following diagram commutes:



These definitions together give us the notion of the **category of presheaves** on X with values in $\mathcal{A}$, denoted $\text{Presheaves}_{\mathcal{A}}(X)$. Here we expanded the definitions as much as possible for clarity, but this definition can be made more concise by saying $\text{Presheaves}_{\mathcal{A}}(X)$ is just the functor category from $\mathcal{U}^{\text{op}}$ to $\mathcal{A}$ where $\mathcal{U}$ is the poset of open sets in X.

Example 4. *Let $\mathcal{A} = \mathbb{R}$-Vect be the category of real vector spaces and X any topological space. Define $C^0(U)$ be the set of all continuous functions from U to $\mathbb{R}$. For $U \subseteq V$ define the morphism $C^0(V) \to C^0(U)$ to be the map which restricts the continuous function $f : U \to \mathbb{R}$ to a continuous function $f|_V : V \to \mathbb{R}$. U^0 is a presheaf on X with values in $\mathbb{R}$-Vect.*

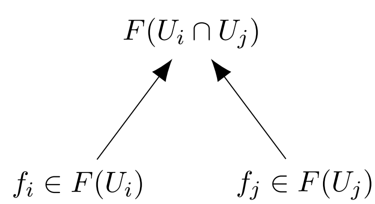
Example 5. *Let M be a smooth manifold (see section 4 for manifold definitions) and let $C^\infty(U)$ be the set of smooth functions $U \to \mathbb{R}$ again using restriction to get morphisms. Then C^∞ is a presheaf on M with values in $\mathbb{R}$-Vect.*

Example 6. *Consider the abelian group A to be a topological space (with the discrete topology), and let X be any topological space. Given an open $U \subseteq X$, let $A(U)$ be the set of continuous maps from U to A. Then A is a presheaf on X.*

Let us now endow our presheaves with some additional structures.

Definition 7. *Let F be a presheaf on X with values in $\mathcal{A}$, and assume that $\mathcal{A}$ is concrete (so objects in $\mathcal{A}$ are just sets possibly with some additional structure). Then F is a* ***Sheaf*** *(on X with values in $\mathcal{A}$) if the following condition holds:*

Let $\{U_i\}$ be an open cover of an open set U in X, and suppose that we have a family $\{f_i \in F(U_i)\}$ such that f_i and f_j get mapped to the same element under the maps



then there exists a unique $f \in F(U)$ which maps to f_i under $F(U) \to F(U_i)$ for each i.

One can check that the three examples given above are all sheaves. Given any presheaf, one can construct a sheaf via a natural rule $a : \text{Presheaves}(X) \to \text{Sheaves}(X)$ in such a way that the functor a is the left adjoint to the forgetful functor $\text{Sheaves}(X) \to \text{Preheaves}(X)$. $a(F)$ is called the **sheafification** of F. See [6] Chapter 1 for details.

Definition 8. *Let $(X, \mathcal{O}_X)$ be a sheaf with values in the category of rings (call this a* ***ringed space****). A* ***coherent sheaf*** *on $(X, \mathcal{O}_X)$ is a sheaf F on X for which:*

- *For each open set $U \subseteq X$, $F(U)$ is a left $\mathcal{O}_X(U)$-module*
- *Given $U \subseteq V$ in X, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_X(V) \times F(V) & \longrightarrow & F(V) \\ \uparrow & & \uparrow \\ \mathcal{O}_X(U) \times F(U) & \longrightarrow & F(U) \end{array}$$

  where the vertical arrows are the sheaf restrictions and the horizontal arrows are the module actions.

- *Every point $x \in X$ has a neighborhood U for which there is a surjective morphism*

$$\Big(\mathcal{O}_X(U)\Big)^{\oplus n} \longrightarrow F(U) \longrightarrow 0$$

  for some natural number n.

- *For any open $U \subseteq X$, any natural number n, and any map of modules*

$$\phi : \Big(\mathcal{O}_X(U)\Big)^{\oplus n} \longrightarrow F(U),$$

  there is a surjective morphism

$$\Big(\mathcal{O}_X(U)\Big)^{\oplus m} \longrightarrow ker\phi \longrightarrow 0$$

  for some natural number m.

The ***category of coherent sheaves*** *on a space X, $\mathrm{Coh}(X)$ is the full subcategory of Presheaves(X) consisting of all coherent sheaves.*

4 Smooth Manifolds and Symplectic Geometry

The objects of the Fukaya category are Lagrangian submanifolds of a given symplectic manifold and morphisms in this category are given by intersections of these submanifolds (more precisely, the morphism sets are exactly the chain groups in Lagrangian intersection Floer cohomology). In this section, we recall the basic notions of smooth manifolds and symplectic geometry needed in what follows. For a more complete treatment, see [5] for smooth manifolds and [3] for symplectic geometry. Here, we will settle for 'working definitions' that are quick to write down and work with, however in a more careful treatment there are better (for example, coordinate invariant) definitions of many of the concepts presented here.

A **smooth manifold** of dimension n is a topological manifold $(M, \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in I})$ such that for each $\alpha, \beta \in I$, $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth function $\mathbb{R}^n \to \mathbb{R}^n$ (i.e. all orders of partial derivatives of coordinate functions exist everywhere).

Define the space of **smooth functions** on M:

$$C^\infty(M) = \{f : M \to \mathbb{R} \mid f \circ \phi_\alpha^{-1} \text{ is smooth for all } \alpha\}.$$

At any point $p \in M$, one can define the **tangent space** at p, denoted T_pM, as the vector space spanned by the functionals $\left\{\frac{\partial}{\partial x^i}\big|_p \;:\; 1 \le i \le n\right\}$ where $\frac{\partial}{\partial x^i}\big|_p : C^\infty(M) \to \mathbb{R}$ acts by $\frac{\partial}{\partial x^i}\big|_p(f) = \left[\frac{\partial}{\partial x^i}\left(f \circ \phi^{-1}\right)\right](p)$. Thus the tangent space is a vector space of dimension equal to the dimension of M, and tangent vectors act on smooth functions by taking their directional derivatives at a point.

The **cotangent space** at $p \in M$ is just the dual space of the tangent space,

$$T_p^*M = (T_pM)^* = \{\text{linear maps } T_pM \to \mathbb{R}\}.$$

The dual basis to $\left\{\frac{\partial}{\partial x^i}\big|_p \;:\; 1 \le i \le n\right\}$ is denoted $\left\{dx^i|_p \;:\; 1 \le i \le n\right\}$ so that $dx^i|_p \frac{\partial}{\partial x^j}\big|_p = \delta_{i,j}$ is equal to 1 when $i = j$ and 0 otherwise.

A **differential k-form** β on M is an assignment of an alternating multilinear map $\beta_p : T_pM^{\oplus k} \to \mathbb{R}$ to each point $p \in M$ which varies smoothly with respect to $p \in M$. Here, alternating means that for any $i < j$,

$$\beta_p(v_1, ..., v_i, ..., v_j, ..., v_k) = -\beta_p(v_1, ..., v_j, ..., v_i, ..., v_k)$$

and multilinear means that β_p is linear in each coordinate.

Given a differential 1-form β, at any $p \in M$, β_p is just an element of the dual space T_p^*M. Thus we can write $\beta_p = \sum_{i=1}^n c_i dx^i|_p$ in terms of the dual basis. Given a differential k-form α and a differential ℓ-form β, define the **wedge product** $\alpha \wedge \beta$ by the formula:

$$(\alpha \wedge \beta)_p(v_1, ..., v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\pi \in S_{k+\ell}} \operatorname{sgn}(\pi)\alpha(v_{\pi(1)}, ..., v_{\pi(k)})\beta(v_{\pi(k+1)}, ..., v_{\pi(k+\ell)}).$$

One can check directly that $\alpha \wedge \beta$ is alternating and multilinear, and is thus a differential $(k + \ell)$-form.

Example 9. *Consider $M = \mathbb{R}^2$ with global coordinates x, y. Then dx and dy are both 1-forms. Their wedge product is:*

$$(dx \wedge dy)|_p(v_1, v_2) = dx|_p(v_1)dy|_p(v_2) - dx|_p(v_2)dy|_p(v_1)$$

for any $v_1, v_2 \in T_p\mathbb{R}^2$.

Lemma 10. *In any coordinate chart* $\phi : U \to \mathbb{R}^n$*, a differential* k*-form* α *can be written as a linear combination of the forms* $dx^{i_1} \wedge ... \wedge dx^{i_k} = dx^I$ *:*

$$\alpha = \sum_I \alpha_I dx^I$$

where $I = \{i_1 < ... < i_k\}$ *and each* α_I *is a map* $U \to \mathbb{R}$.

Define the **exterior derivative** $d\alpha$ of the form α locally via the formula

$$d\alpha = \sum_I d\alpha_I \wedge dx^I$$

where the exterior derivative df of a function $f : U \to \mathbb{R}$ is defined by:

$$df(v) = v(f).$$

One should check that this definition is independent of the coordinate chart used.

Definition 11. *A* ***symplectic form*** ω *on* M *is a differential 2-form which satisfies the following properties:*

- $d\omega = 0$
- *For any vector* $v \neq 0$ *there exists a vector* u *for which* $\omega(v, u) \neq 0$.

These conditions are usually stated in words as "ω is **closed** and **nondegenerate**". Given a smooth manifold M with a symplectic form ω, we will refer to the pair (M, ω) as a **symplectic manifold**. One can check that all symplectic manifolds are even dimensional.

Example 12. *Consider the manifold* $M = \mathbb{R}^{2n}$ *with global coordinates* $\{x^1, ..., x^n, y^1, ..., y^n\}$*. Consider the* ***standard symplectic form****:*

$$\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i.$$

The reader should check that ω *is a symplectic form on* $\mathbb{R}^{2n}$*. In a certain sense, by Darboux's theorem, every symplectic form on any manifold is locally equivalent to this form (See* [3] *for a more precise explanation).*

Given a submanifold $L \subseteq M$ (a subset of M which is a smooth manifold), the tangent space T_pL is a subspace of T_pM for each $p \in L$.

Definition 13. *Let* (M, ω) *be a symplectic manifold. A submanifold* L *is a* ***lagrangian submanifold*** *if*

- $\dim L = \frac{1}{2} \dim M$
- $\omega_p|_{T_pL} = 0$ *for each* $p \in L$.

Example 14. *Continuing on from the previous example, we see that the submanifolds*

$$M_x = \{(x^1, ..., x^n, 0, ..., 0) \mid x^i \in \mathbb{R}\}$$

$$M_y = \{(0, ..., 0, y^1, ..., y^n) \mid y^i \in \mathbb{R}\}$$

are lagrangian submanifolds with respect to the standard symplectic form.

5 Lagrangian Intersection Floer Theory

To construct the Fukaya category, we will need to make use of a powerful construction on symplectic manifolds known as **Lagrangian intersection Floer Cohomology**. Here we will say just enough to be able to define the Fukaya category and leave out most of the technical details.

The **tangent bundle** to M is the space $TM = \amalg_{p \in M} T_pM$. It is a good exercise to write down charts based on the atlas for M which gives TM the structure of a smooth manifold. To construct the Lagrangian intersection Floer chain complex, we will need our manifold M to have an **almost complex structure** J, compatible with the symplectic form ω. That is, a map $J : TM \to TM$ which satisfies

1. $J^2 = -1$
2. $\omega(Ju, Jv) = \omega(u, v)$. That is, ω is J invariant.
3. $g_J(u, v) := \omega(u, Jv)$ is a symmetric positive definite bilinear form.

Example 15. *Let* $M = \mathbb{R}^{2n}$ *and define the standard almost complex structure*

$$J_0\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial}{\partial y^j}\bigg|_p$$

$$J_0\left(\frac{\partial}{\partial y^j}\bigg|_p\right) = -\frac{\partial}{\partial x^j}\bigg|_p$$

The reader should check that J_0 *is compatible with the standard symplectic form* ω_0 *from the previous section.*

It is a classical result that every symplectic manifold has compatible almost complex structures, and the space of all such structures is contractible. In the statement of homological mirror symmetry, we will need the concept of a **Kähler manifold**. A Kähler manifold is just a symplectic manifold with a compatible **integrable** almost complex structure. Since we will not do much with these manifolds, we skip the definition of integrable and instead refer the reader to [3].

Given a symplectic manifold M with a compatible almost complex structure J, suppose we have two transversally intersecting Lagrangian submanifolds L_1, L_2 (that is, $T_pL_1 + T_pL_2 = T_pM$ at any $p \in L_0 \cap L_1$). Actually, even if the submanifolds do not intersect transversally, one can perturb them so that they do, and the following theory can be shown to be invariant under these small pertubations (see [2] for details). We will now glance at the answer of what the Lagrangian intersection Floer chain complex is, and afterwards describe all of the pieces which we have not yet introduced. The Lagrangian intersection Floer chain complex is

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p$$

endowed with the differential

$$d(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\mathrm{ind}[u]=1}} \Big(\#\mathcal{M}\big(p, q; [u]\big)\Big) T^{E(u)} q.$$

Let us now clear up the mystery of what this all means. The reader is owed the following definitions:

1. The **Novikov field** Λ over a field $\mathbb{K}$ (for our purposes we will use $\mathbb{K} = \mathbb{Z}_2$):

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

2. The **moduli space** $M\big(p, q; [u]\big)$:

   Given $p, q \in L_0 \cap L_1$. A **holomorphic strip** between p and q is a map $u : \mathbb{R}_s \times [0,1]_t$ (subscripts indicating which variable is used on which piece) satisfying the following conditions:

   - The Cauchy Riemann equation: $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$.
   - Boundary conditions:

$$u(s, 0) \in L_0, \qquad u(s, 1) \in L_1, \qquad \lim_{s \to -\infty} u(s, t) = q, \qquad \lim_{s \to \infty} u(s, t) = p.$$

   - Finite energy:

$$E(u) := \iint \left|\frac{\partial u}{\partial s}\right| ds\ dt < \infty.$$

   Call $E(u)$ the **energy** of the holomorphic strip u. One can show that this energy depends only on the class $[u] \in \pi_2(M, L_0 \cup L_1)$ in the 2nd relative homotopy group.

   Remark: for this to make sense, notice that by the Riemann mapping theorem, $\mathbb{R} \times [0,1]$ is biholomorphic to the closed unit disk with two points removed $\overline{D^2} \setminus \{\pm 1\}$, and u extends to a map $\overline{D^2} \to M$ with $u(-1) = p$ and $u(1) = q$. So when we talk about $[u]$ as an element of $\pi_2(M, L_0 \cup L_1)$ we need to think of u in this way as a map from the disk into M (see figure 1).

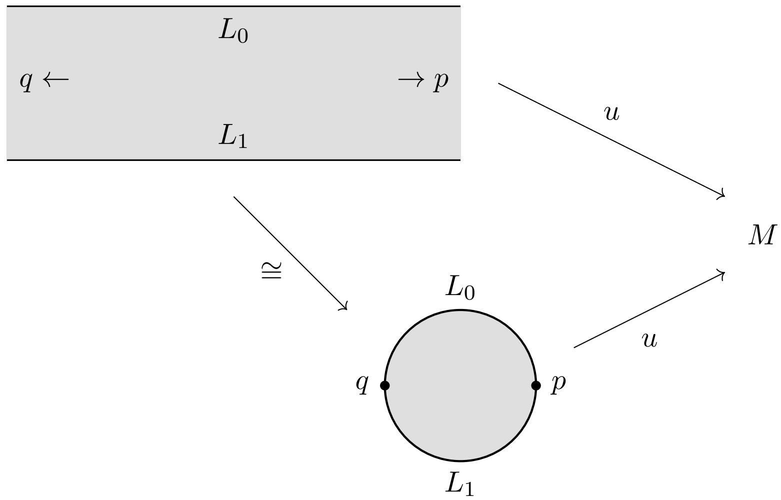


Figure 1: We can think of a holomorphic strip equivalently as a map from a closed unit disk. Labels on boundaries indicate where they are sent in M via the map u.

Now, let $\mathcal{W}(p,q)$ be the set of finite energy holomorphic strips between p and q. $\mathbb{R}$ acts on $W(p,q)$ via translation in the s direction: $(a \cdot u)(s,t) = u(s-a,t)$ for $a \in \mathbb{R}$. We set

$$\mathcal{M}(p,q) = \mathcal{W}(p,q) / \mathbb{R}$$

$$\mathcal{W}(p,q;[u]) = \{v \in \mathcal{W}(p,q) \mid [v] = [u]\},$$

$$\mathcal{M}(p,q;[u]) = \mathcal{W}(p,q;[u]) / \mathbb{R}$$

where again $[u]$ denotes the class of u in $\pi_2(M, L_0 \cup L_1)$. The moduli space $\mathcal{W}(p,q;[u])$ turns out to be a smooth manifold.

3) The **Maslov index** $\mathrm{ind}[u]$, defined by:

$$\mathrm{ind}[u] = \dim \mathcal{M}(p,q;[u]) + 1.$$

When $\mathrm{ind}[u] = 1$ the moduli space $\mathcal{M}(p,q;[u])$ has dimension 0, so by compactness, this is a finite set of points. We will work over $\mathbb{K} = \mathbb{Z}_2$ and let $\#\mathcal{M}(p,q;[u])$ just be the number of points in the moduli space modulo 2.

Details aside, here is the point: the Maslov index can be used to define degrees of points in $L_0 \cap L_1$ in such a way that the dimensions of the moduli spaces $\mathcal{M}(p,q;[u])$ are given by the differences in degrees of p and q. Under suitable conditions, the Lagrangian intersection Floer complex is indeed a chain complex with a degree 1 differential. One condition one can use to guarantee this is $[\omega] \cdot \pi_2(M, L_i) = 0$ for $i = 0, 1$ (where the action is defined by integrating ω on any disk bounded by L_i). For details, see [2].

6 A_∞-Categories and the Fukaya Category

The Fukaya category has the structure of an A_∞-category. First we define A_∞-categories in general and then describe the Fukaya category as an example.

Definition 16. *An A_∞-category consists of objects L_i and morphism spaces $Hom(L_i, L_j)$ which are graded vector spaces such that for each $k \geq 1$ and each tuple $(L_{i_0}, L_{i_1}, ..., L_{i_k})$, we have a degree $(2-k)$ operation:*

$$m_k : Hom(L_{i_{k-1}}, L_{i_k}) \otimes Hom(L_{i_{k-2}}, L_{i_{k-1}}) \otimes ... \otimes Hom(L_{i_0}, L_{i_1}) \longrightarrow Hom(L_{i_0}, L_{i_k})$$

subject to the A_∞ relations:

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} m_{k+1-\ell}\Big(p_k, ..., p_{j+\ell+1}, m_\ell\big(p_{j+\ell}, ..., p_{j+1}\big), p_j, ..., p_1\Big) = 0.$$

(using commas instead of tensor symbols for brevity) whenever the above sum is defined.

This sum looks a bit scary; but after looking at the relations for small k we shall see it is a natural constraint:

$\boxed{k=1}$ The A_∞ relation reads

$$m_1\Big(m_1(p_1)\Big) = 0$$

or in other words, m_1 is a differential.

$\boxed{k=2}$ The A_∞ relation reads

$$m_1\Big(m_2(p_2, p_1)\Big) + m_2\Big(p_2, m_1(p_1)\Big) + m_2\Big(m_1(p_2), p_1\Big) = 0.$$

Considering $m_1 = d$ as a differential and $m_2 = \cdot$ as a product, this reads

$$-d(p_2 \cdot p_1) = p_2 \cdot d(p_1) + d(p_2) \cdot p_1.$$

Up to a sign, this looks like a Liebniz rule for the product $m_2 = \cdot$ with respect to the differential $m_1 = d$.

$\boxed{k=3}$ The A_∞ relation reads

$$m_2\Big(p_3, m_2(p_2, p_1)\Big) + m_2\Big(m_2(p_3, p_2), p_1\Big) +$$
$$m_1\Big(m_3(p_3, p_2, p_1)\Big) + m_3\Big(p_3, p_2, m_1(p_1)\Big) + m_3\Big(p_3, m_1(p_2), p_1\Big) + m_3\Big(m_1(p_3), p_2, p_1\Big) = 0$$

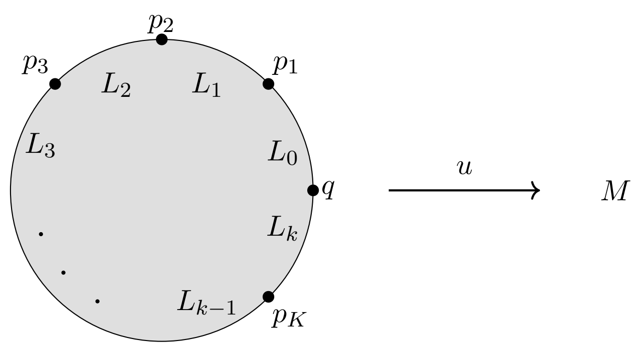
Again, viewing m_2 as a product, this looks like the requirement that m_2 is associative up to some error terms (and up to a sign).

Definition 17. *Given a symplectic manifold* (M, ω)*, the* ***Fukaya category****,* $\mathrm{Fuk}(M)$ *is the* A_∞ *category with*

- *Objects: Compact closed oriented spin Lagrangians* L *with* $[\omega] \cdot \pi_2(M, L) = 0$*.*
- *Morphisms:* $Hom(L_0, L_1) := CF(L_0, L_1)$
- $m_k : CF(L_{i_{k-1}}, L_{i_k}) \otimes CF(L_{i_{k-2}}, L_{i_{k-1}}) \otimes \ldots \otimes CF(L_{i_0}, L_{i_1}) \longrightarrow CF(L_{i_0}, L_{i_k})$ *is defined by*

$$m_k(p_k, p_{k-1}, \ldots, p_1) = \sum_{\substack{q \in L_{i_0} \cap L_{i_k} \\ [u]: ind[u] = 2-k}} \#\mathcal{M}(p_1, \ldots, p_k; [u]) T^{E(u)} q$$

where $\mathcal{M}(p_1, \ldots, p_k; [u])$ *is the moduli space of finite energy holomorphic* k*-gons. That is, maps* $u : \bar{D}^2 \to M$*:*



where the labeling of points and arcs correspond to where they are sent by u*.*

It takes a fair amount of work analyzing compactifications of moduli spaces to show that these indeed satisfy the A_∞ relations. We will not do that here (see [2] for details).

7 Mirror Symmetry and Closing Remarks

We now have all of the pieces in place we need to understand Kontsevitch's conjecture:

Conjecture 18 (Kontsevitch)**.** *Certain Kähler manifolds come in mirror pairs* $X, X^\vee$ *such that there is an equivalence*

$$D^b\mathrm{Coh}(X) \cong D^b\mathrm{Fuk}(X^\vee)$$

between the bounded derived category of coherent sheaves on X *and the bounded derived Fukaya category on* $X^\vee$*.*

To be fair to the reader, it should be noted that we are still missing some details:

1. The Fukaya category is not a true category (associativity holds only up to homotopy in an A_∞-category) so one needs to be careful when defining what it means to take the derived category of an A_∞-category.

2. We have not defined what it means for categories to be equivalent.

3. Derived categories are examples of what are known as **triangulated categories** so the above is actually a triangulated equivalence.

After learning the drastically differing details of the constructions of these categories $D^b\mathrm{Coh}(X)$ and $D^b\mathrm{Fuk}(X^\vee)$, it should be of great surprise and delight that there should be any kind of equivalence. For any mirror pair of manifolds, one can now use sheaf theoretic methods to compute Fukaya categories. Conversely, one can use methods of symplectic geometry to compute the categories of coherent sheaves. This symmetry may lead to exciting new advances in both fields. Of course, this paper has left out a large amount of details. The interested reader should consult [1] for further study into mirror symmetry.

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