A Categorification of the Laguerre Polynomials

Alex Chandler

North Carolina State University

May 7, 2018

Orthogonal Polynomials

The polynomial ring $\mathbb{R}[x]$ has standard basis $\{1, x, x^2, \ldots\}$ orthogonal with respect to inner product

$$(f, g) = \sum_{n \geq 0} \left([x^n]f\right) \left([x^n]g\right)$$

By changing the inner product we can get other families of orthogonal polynomials.

Orthogonal Polynomials

We consider the inner product

$$(f, g) = \int_a^b f(x)g(x)w(x)\mathrm{d}x$$

for some choice of weight function $w(x)$ and interval $[a, b]$.
Apply Gram-Schmidt to the standard basis w.r.t. this inner product to get a family of orthogonal polynomials.

$w(x)$	e^{-x^2}	$\sqrt{1-x^2}$	e^{-x}
(a, b)	$(-\infty, \infty)$	$(-1, 1)$	$(0, \infty)$
Family	Hermite	Chebyshev	Laguerre

Laguerre Polynomials

Consider the choice:

$$(f,g) = \int_0^\infty f(x)g(x)e^{-x}\mathrm{d}x.$$

Applying Gram-Schmidt we get the Laguerre polynomials

$$L_n(x) = \sum_{k\geq 0}(-1)^k \binom{n}{k}\frac{x^k}{k!}$$

Decategorification

Def: The Grothendieck group $K_0(\mathcal{A})$ of (additive) category $\mathcal{A}$

- Free Abel. group generated by symbols $[M]$ for $M \in \mathrm{Ob}(\mathcal{A})$
- Subject to relations $[M] = [N] + [K]$ whenever $M \cong N \oplus K$
- For example $K_0(\Bbbk\text{-Vect}) \cong \mathbb{Z}$ generated by $[\Bbbk]$

Theorem

$K_0(\mathcal{C}(\mathcal{A})) \cong K_0(\mathcal{A})$ canonically via the map

$$[\dots M_n \to M_{n-1} \to M_{n-2} \dots] \mapsto \sum_{k \geq 0} (-1)^k [M_k]$$

Categorification

Idea: "lift" elements of a set to objects of a category thus gaining extra structure.

Example: To categorify some element g in an abelian group G:

- Construct a category $\mathcal{C}$ with $K_0(\mathcal{C}) \cong G$
- Find an object $M_g \in \mathrm{Ob}(\mathcal{C})$ such that $[M_g] \mapsto g$ under this isomorphism (make identification $[M_g] = g$)

Question: What can we learn about g by looking at M_g?

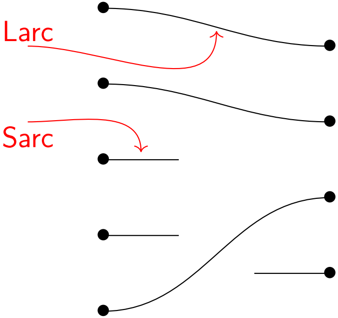
Categorifying a Polynomial $f(x) \in \mathbb{Z}[x]$

- Category $\mathcal{C}$ with $K_0(\mathcal{C}) \cong \mathbb{Z}[x]$
- Family of objects P_n with $[P_n] = x^n$
- Object M with $[M] = f(x)$
- Functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ so that $[P_n \otimes P_m] = [P_{n+m}]$ categorifying $x^n x^m = x^{n+m}$

In 2011, Sazdanovic and Khovanov constructed such a category $\mathcal{C}$ via the "SLarc" algebra.

Def An (n, m)-SLarc diagram has

- n left endpoints and m right endpoints
- Each endpoint has a short arc (sarc) or a long arc (larc)
- No crossings, returns, or critical points



$(5, 4)$-SLarc diagram

Width=3

Def Let ${}_nB_m$ denote the set of (n, m)-SLarc diagrams. Then

$$|{}_nB_m| = \sum_{k\geq 0} \binom{n}{k}\binom{m}{k} = \binom{n+m}{n}$$

width

Define

$$B = \bigsqcup_{n,m\geq 0} {}_nB_m$$

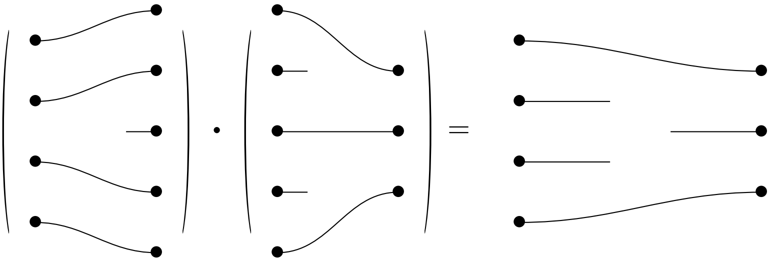
and define A as the $\Bbbk$-vector space with basis B:

$$A = \Bbbk B$$

Algebra of SLarcs

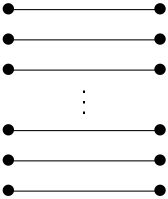
Algebra structure on $A = \Bbbk B$:

- Multiply compatible diagrams by horizontal concatenation
- Incompatible diagrams have product 0
- Floating arcs are set to 0:  $= 0$



Orthogonal Idempotents

Def: Let $1_n \in {}_nB_n$ be the unique diagram with no sarcs.

$1_n =$ 

The elements $\{1_n \mid n \in \mathbb{N}\}$ are mutually orthogonal idempotents:

$$1_n 1_n = 1_n$$

$$1_n 1_m = 0 \quad (n \neq m)$$

- **Def:** Let M be an A-module. M is projective if $M \oplus N$ is free for some A-module N.
- Let A-pmod denote the category of f.g. projective A-modules.
- The modules $P_n := A \cdot 1_n$ are indecomposable projectives
- $\mathrm{Hom}(P_n, P_m) = \Bbbk \cdot {}_nB_m$

Theorem (Sazdanovic, Khovanov, 2011)

Every f.g. projective A-module can be written as a direct sum of P_n's.

Theorem (Sazdanovic, Khovanov, 2011)

The map $K_0(A\text{-pmod}) \to \mathbb{Z}[x]$ sending $[P_n] \mapsto x^n$ is a group isomorphism.

Standard Modules

- $K_n \leq P_n$ submodule generated by diagrams with at least one right sarc
- Standard module $M_n = {}^{P_n}\!/_{K_n}$ has a basis of diagrams with no right sarcs

Theorem (Sazdanovic, Khovanov, 2011)

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k]$$

and

$$[P_n] = \sum_{k=0}^{n} \binom{n}{k} [M_k]$$

Under the isomorphism $[P_n] \mapsto x^n$ we find that

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} x^k = (x-1)^n$$

so M_n categorifies $(x-1)^n$. The other equality gives us the relation

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k$$

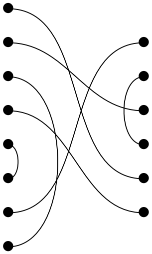
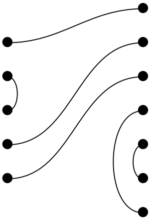
a non-obvious fact we get for free

Categorifying Orthogonal Polynomials in $\mathbb{Z}[x]$

Idea: Categorify the bilinear form $(\cdot,\cdot)$ on $\mathbb{Z}[x]$. That is, we need:

- Diagrammatic algebra A
- Category $\mathcal{A} = A$-pmod with $K_0(\mathcal{A}) \cong \mathbb{Z}[x]$ as groups via $[P_n] \mapsto x^n$
- $(x^n, x^m) = \dim \operatorname{Hom}(P_n, P_m) = |{}_nB_m|$
  That is, construct a diagrammatic algebra where ${}_nB_m$ is counted by (x^n, x^m).

Categorifying Hermite and Chebyshev Polynomials

	Hermite	Chebyshev
Formula	$\sum(-1)^k \binom{n}{n-2k}(2k-1)!!x^{n-2k}$	$\sum(-1)^k \binom{n-k}{k} x^{n-2k}$
(x^n, x^m)	$(n+m-1)!!$	Catalan C_k, $k = \frac{n+m}{2}$
Diagrams	Allow returns, crossings, no sarcs	non-crossing matchings
		

Categorifying Laguerre Polynomials

	Laguerre
Formula	$L_n(x) = \sum(-1)^k \binom{n}{k} \frac{x^k}{k!}$
(x^n, x^m)	$(n+m)!$
Diagrams	(n, m)-Permutation Digraphs

Problem! Coeff of x^k in $L_n(x)$ is not an integer...
A solution: View $L_n(x)$ as an *exponential polynomial*

$$L_n(x) \in \mathbb{Z}^{\text{exp}}[x] := \mathbb{Z}\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots\right]$$

so the coeff of $\frac{x^k}{k!}$ in $L_n(x)$ is $(-1)^k \binom{n}{k}$.

Categorifying Laguerre Polynomials

Strategy: Find a category $\mathcal{C}$ with

- $G_0(\mathcal{C}) \cong \mathbb{Z}^{\exp}[x]$, $[P_n] \mapsto \frac{x^n}{n!}$
  (easy: since $\mathbb{Z}[x] \cong \mathbb{Z}^{\exp}[x]$ as groups, $\mathcal{C} = A$-pmod works
- $(\frac{x^n}{n!}, \frac{x^m}{m!}) = \dim \operatorname{Hom}(P_n, P_m)$
- Find modules M_n for which $[M_n] = L_n(x)$

Choosing Diagrammatics

- Bilinear form for Laguerre is $(f,g) = \int_0^\infty f(x)g(x)e^{-x}\mathrm{d}x$
- $(\frac{x^n}{n!}, \frac{x^m}{m!}) = \int_0^\infty \frac{x^n}{n!}\frac{x^m}{m!}e^{-x}\mathrm{d}x = \frac{(n+m)!}{n!m!} = \binom{n+m}{n}$
- recall for SLarc algebra A, $\dim \mathrm{Hom}(P_n, P_m) = \binom{n+m}{n}$
- So $\mathcal{C} = A$-pmod is the right category to look at

Choosing Modules

Recall the standard modules M_n satisfy

$$[M_n] = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} [P_k]$$

Define the *signed standard modules*

$$N_n = M_n[n] = 0 \to (M_n)_{(n)} \to 0$$

so in the Grothendieck group

$$[N_n] = (-1)^n [M_n]$$

Choosing Modules

Theorem (C. 2017)

$K_0(A\text{-pmod}) \cong \mathbb{Z}^{exp}[x]$ via $[P_n] \mapsto \frac{x^n}{n!}$ and signed standard modules categorify Laguerre polynomials under this isomorphism

$$[N_n] = L_n(x)$$

What's Next?

- Categorify product $\frac{x^n}{n!} \cdot \frac{x^m}{m!} = \binom{n+m}{n} \frac{x^{n+m}}{(n+m)!}$
- Need a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- Set $P_n \otimes P_m = P_{n+m}^{\binom{[n+m]}{n}}$
- Need to describe action on morphisms. That is, given:

$$f : P_n \to P_k$$

$$g : P_m \to P_\ell,$$

we need to define

$$f \otimes g : P_n \otimes P_m \to P_k \otimes P_\ell$$

Thank you!