Chapter 1
Categorical Diagonalization

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Abstract Beginning with the scenario in which a linear operator is diagonalized, we develop refined notions of the involved properties—the vanishing of a polynomial in the endofunctor—and structures—the acquiring of an idempotent decomposition of the space the operator acts on—and discuss how to go between the two. Most importantly, we recall Lagrange interpolation, which, from a collection of distinct eigenvalues, produces an idempotent decomposition. We then categorify this picture by finding categorified decompositions corresponding to functors on monoidal homotopy categories. We then proceed to look at several examples, culminating in the diagonalizability of the full twist functor on Rouqier Complexes in the representation theory of Hecke algebras.

1.1 Classical Linear Algebra

Recall the following situation from linear algebra. Suppose we have a linear endomorphism

$$f : V \to V$$

of $\mathbb{k}$-vector spaces with $\dim V = n$. Suppose furthermoe that there are distinct scalars $\{\kappa_i\}_{i=1}^n \subset \mathbb{k}$ and vectors $\{\mathbf{x}_i\}_{i=1}^n \subset V$ such that for all $i \in \{1, \dots, n\}$, we have

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$$f\mathbf{x}_i = \kappa_i \mathbf{x}_i.$$

We called the κ_i's eigenvalues and the $\mathbf{x}_i$'s eigenvectors. If we choose the list of eigenvectors as the basis for V in both domain and codomain of f, then the corresponding matrix for f is a diagonal matrix

$$\begin{bmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_n \end{bmatrix}.$$

We called this matrix the diagonalization of f.

Remark 1.1. Note that distinct eigenvalues is a sufficient but not necessary condition for diagonalizability of a linear endomorphism. The case where an operator is diagonalized with repeated eigenvalues however is more difficult to categorify, and hence we will restrict our attention to the case where all eigenvalues are distinct.

This allows us to deduce the relation

$$(f-\kappa_1)(f-\kappa_2)\cdots(f-\kappa_n) = 0.$$

The property of satisfying such an equation is called prediagonalizability. It will be useful for our purposes—the categorification of this situation—to formulate further aspects of f by investigating the relevant arrow theoretic structure. Consider the direct sum decomposition given by the eigenvector basis.

$$V = \bigoplus_{i=1}^{n} \Bbbk \mathbf{x}_i.$$

Universal properties endow each direct summand with projections and inclusions:

$$V \to \Bbbk\mathbf{x}_i \qquad \Bbbk\mathbf{x}_i \to V.$$

Note that one composition $\Bbbk\mathbf{x}_i \to V \to \Bbbk\mathbf{x}_i$ of these maps is the identity for all i. We will be more interested in the other composition $V \to \Bbbk\mathbf{x}_i \to V$, which we denote $p_i : V \to V$. We may call the p_i *projectors*. The set $\{p_i\}_{i=1}^n$ gives a useful decomposition of V. More formally, one can check that it satisfies the following relations.

$$p_i^2 = p_i \qquad p_i p_j = 0 \text{ for } i \neq j \qquad \textstyle\sum_i p_i = \mathbb{1}_V$$

Intuitively, the p_i's pick out the direct summand corresponding to the κ_i eigenspace. This lets us recast $(f-\kappa_i)\mathbf{x}_i = 0$ more arrow theoretically:

$$(f-\kappa_i)p_i = 0 = p_i(f-\kappa_i).$$

We can abstract this situation as follows. For the remainder of the section, let R be a commutative ring, A an R-algebra, and V an A-module. Above we simply had

R a field $\Bbbk$, V a $\Bbbk$-vector space, and $A = \operatorname{End} V$. We now formalize the ideas above in the following definitions.

Definition 1.1. We say $f \in A$ is *prediagonalizable* if there exist distinct scalars $\{\kappa\}_{i\in I} \subset R$ such that

$$\prod_{i\in I}(f - \kappa_i) = 0.$$

We also assume that any product over a strict subset S of I is nonzero.

Definition 1.2. A finite family $\{p_i\}_{i\in I} \subset A$ is said to be a *complete orthogonal idempotent decomposition* of A if $p_i^2 = p_i$, $p_i p_j = 0$ when $i \neq j$, and $\sum_i p_i = 1$.

Definition 1.3. We say that $f \in A$ is *diagonalizable* if there exists a complete orthogonal idempotent decomposition $\{p_i\}_{i\in I}$ of A and a family $\{\kappa_i\}_{i\in I} \subset R$ such that the following condition holds for all $i \in I$.

$$(f - \kappa_i)p_i = 0 = p_i(f - \kappa_i).$$

When none of the p_i are zero, we call $\{\kappa_i\}_{i\in I}$ a *spectrum*.

Under a small assumption, these two conditions are equivalent.

Proposition 1.1. *If $f \in A$ is diagonalizable with spectrum $\{\kappa_i\}_{i\in I}$, it is prediagonalizable. If $f \in A$ is prediagonalizable with scalars $\{\kappa_i\}_{i\in I}$ such that $\kappa_i - \kappa_j$ is invertible in A whenever $i \neq j$, then f is diagonalizable.*

Proof. Suppose f is diagonalizable with spectrum $\{\kappa_i\}_{i\in I}$ and complete orthogonal idempotent decomposition $\{p_i\}_{i\in I}$. Then

$$\begin{aligned}
\prod_{i\in I}(f - \kappa_i) &= \sum_{j\in I} p_j \prod_{i\in I}(f - \kappa_i) \\
&= \sum_{j\in I} p_j(f - \kappa_j)\prod_{i\neq j}(f - \kappa_i) \\
&= 0
\end{aligned}$$

The spectrum condition ensures that no subproduct vanishes. Now, conversely, suppose that f is prediagonalizable. We construct the idempotents p_i via *Lagrange interpolation*:

$$p_i(f) = \prod_{j\neq i} \frac{f - \kappa_j}{\kappa_i - \kappa_j}$$

This is well defined since $\kappa_i - \kappa_j$ is by assumption a unit for $i \neq j$. It follows immediately that $(f - \kappa_i)p_i(f) = 0 = p_i(f)(f - \kappa_i)$. It remains to show that $\{p_i(f)\}_{i\in I}$ constitutes a complete orthogonal idempotent decomposition.

By prediagonalizability, the polynomial ring $\Bbbk[f] \cong \Bbbk[x]/\langle (f - \kappa_1)\cdots(f - \kappa_{|I|})\rangle$ consists of representative polynomials of degree strictly below $|I|$. We have that $p_i(f) \in \Bbbk[f]$ satisfies $p_i(\kappa_i) = 1$ and $p_i(\kappa_j) = 0$ when $i \neq j$. Hence $L(f) = \sum_i a_i p_i(f)$ is the unique polynomial in $\Bbbk[f]$ such that $L(\kappa_i) = a_i$ for all i. It follows that any

$Q(f) \in \Bbbk[f]$ can be rewritten as $\sum_i Q(\kappa_i) p_i(f)$. In particular, letting $Q(f) = 1$ yields completeness $1 = \sum_i p_i(f)$, letting $Q(f) = p_i(f)^2$ yields idempotence $p_i(f)^2 = p_i(f)$, and letting $Q(f) = p_i(f)p_j(f)$ yields orthogonality $p_i p_j = 0$ when $i \neq j$.

The following section will be dedicated to categorifying this construction.

1.2 Categorified Linear Algebra

We want to categorify everything in sight. We replace the commutative ring R with an R-additive category $\mathscr{R}$, the R-algebra A with an R-additive monoidal category $\mathscr{A}$, and the A-module V with an $\mathscr{A}$-module $\mathscr{V}$, i.e. an R-additive category $\mathscr{V}$ equipped with a monoidal functor $\mathscr{A} \to \mathrm{End}(\mathscr{V})$. Finally, we fix a monoidal subcategory $\mathscr{K} \subset \mathscr{A}$ of *scalar objects*, which induce *scalar endofunctors* $\lambda \otimes - : \mathscr{V} \to \mathscr{V}$ for each scalar λ. We call a scalar λ *invertible* if there is another scalar λ^{-1} such that $\lambda \otimes \lambda^{-1} \simeq \mathbb{1} \simeq \lambda^{-1} \otimes \lambda$. We call a scalar λ *small* if the infinite direct sum $\bigoplus_{n \geq 0} \lambda^{\otimes n}$ exists in $\mathscr{A}$ and is isomorphic to its direct product. The smallness condition will be needed to ensure the well definedness of the infinite constructions we will consider.

Remark 1.2. If you are so inclined, you can think of A as a monoid object on R in the monoidal category of commutative rings under tensor product, and $\mathscr{A}$ as a (weak) monoid object on $\mathscr{R}$ in a suitable monoidal 2-category of R-additive categories.

Remark 1.3. Instead of an R-additive category, we may also consider an Abelian or Triangulated category, in which case we restrict $\mathrm{End}(\mathscr{V})$ to exact endofunctors.

What would it mean to categorify the equation $fx_i = \kappa_i x_i$? A naive guess would be to consider $\mathscr{V}$-objects M such that, given scalar λ and $F \in \mathscr{V}$, there is an isomorphism $F \otimes M \cong \lambda \otimes M$. We call such M *weak λ-eigenobjects* for F. For such a structure to be sufficiently amenable to categorical machinery, however, we must fix this isomorphism in a natural way.

Definition 1.4. Suppose $\lambda \in \mathscr{K}$, $\alpha : \lambda \to F$ in $\mathscr{A}$, and $M \in \mathscr{V}$ is nonzero. We call M an *eigenobject of F with eigenmap* α or an *α-eigenobject* if the following is an isomorphism.

$$\alpha \otimes \mathbb{1}_M : \lambda \otimes M \to F \otimes M.$$

In this situation, we call α a *forward eigenmap*. We may also consider a *backward eigenmap* $\beta : F \to \lambda$ defined similarly. We define the *α-eigencategory* $\mathscr{V}_\alpha$ to be the smallest full additive subcategory of $\mathscr{V}$ containing the α-eigenobjects.

We now want to consider categorified analogs of prediagonaizability and diagonalizability. To do so, we must categorify the difference $f - \kappa_i$. This can be done via the mapping cone construction. If one is not familiar with this choice, intuition can be gained from considering the fact that the cone of an isomorphism $\varphi : N \to N'$ is contractible: $\mathrm{Cone}(\varphi) \simeq 0$ is a deformation retract. Because we will work in homotopy categories, we will be allowed to "cancel" isomorphisms of objects, thus categorifying $n-n=0$. In order to ensure the existence of mapping cones, we switch to their natural setting in triangulated categories, in which we have canonical distinguished triangles

$$X \xrightarrow{\phi} X' \to \mathrm{Cone}(\phi).$$

The Grothendieck group of a triangulated category is given relations $\langle C_1 \rangle + \langle C_3 \rangle = \langle C_2 \rangle$ for distinguished triangles $C_1 \to C_2 \to C_3$, thus yielding the relation

$$\langle \mathrm{Cone}(\phi) \rangle = \langle X' \rangle - \langle X \rangle,$$

and hence successfully categorifying subtraction. More formally, we will henceforth let $\mathscr{A}$ be a *monoidal homotopy category* $(\mathscr{A}, \otimes, \mathbb{1})$, i.e. we assume that there exists an additive monoidal category $(\mathscr{B}, \otimes, \mathbb{1})$ such that $\mathscr{A}$ is a full triangulated subcategory of $\mathrm{K}(\mathscr{B})$, the homotopy category of chain complexes in $\mathscr{B}$, with its induced tensor product. With this in hand, we can now make sense of how to categorify prediagonalizability.

Definition 1.5. We say $F \in \mathscr{A}$ is *categorically prediagonalizable* if there is a finite set of maps $\{\alpha_i : \lambda_i \to F\}_{i \in I}$, with λ_i scalar objects, such that

$$\bigotimes_{i \in I} \mathrm{Cone}(\alpha_i) \simeq 0$$

is a minimal vanishing tensor product. We call the set $\{\alpha_i\}_{i \in I}$ a *prespectrum* for F. Note that it is not unique.

Remark 1.4. We will also like an extra condition called *strong commutativity*, which ensures that any permutation $\pi : I \to I$ preserves the vanishing of the tensor product of eigencones

Just as in the decategorified case, we would like to use prediagonalizability to construct an idempotent decomposition via a sort of Lagrange interpolation. We first need to define what that would entail. To do so, however, we will introduce the language of twisted complexes. The goal of twisted complexes is to formalize iterated mapping cones. This will entail interesting categorifications of certain sums.

Definition 1.6. A *twisted complex* $(C_i, d_{ij})_{i,j \in I}$ consists of poset-indexed chain complexes (in some additive category) C_i, along with maps $d_{ij} : C_j \to C_i$ for $j \leq i$ such that $d_{ii} = d_{C_i}$. We assume $(I, \leq)$ is *interval finite*, i.e. that $[j,i] = \{k | j \leq k \leq i\}$ is finite, and furthermore stipulate that $\sum_{k \in [i,j]} d_{jk} d_{ki} = 0$.

Twisted complexes have as special cases bicomplexes. Given a twisted complex, we can obtain a chain complex called the convolution, which generalizes the notion of a total complex of a bicomplex.

Definition 1.7. Given a twisted complex $(C_i, d_{ij})_{i,j \in I}$, its *convolution* or *total complex* is a chain complex given by

$$\mathrm{Tot}(C_i, d_{ij})_{i,j\in I} = \left(\bigoplus_{i\in I} C_i, \sum_{(i,j)\in I^2} d_{ij} \right).$$

The convolution is an interesting categorification of the sum of complexes. We can then use this to provide an interesting categorification of our notion of idempotent decomposition.

Remark 1.5. Note that the convolution is distinct from simply taking the direct sum of complexes (C_i, d_{C_i}) in that it provides a more interesting "twisted" differential.

Definition 1.8. Let $(\mathscr{A}, \otimes, \mathbb{1})$ be a monoidal homotopy category and $\mathbf{P} = (\mathbf{P}_i, d_{ij})_{i,j\in I}$ a finitely indexed twisted complex in $\mathscr{A}$. We say $\mathbf{P}$ is an *idempotent decomposition* of $\mathbb{1} \in \mathscr{A}$ if the P_i are nonzero, $\mathbf{P}_i \otimes \mathbf{P}_i \simeq \mathbf{P}_i$, $\mathbf{P}_i \otimes \mathbf{P}_j \simeq 0$ when $i \neq j$, and $\mathrm{Tot}(\mathbf{P}) \simeq \mathbb{1}$.

This gives us the language to define what it means for a functor $F \in \mathscr{A}$ to be diagonalizable.

Definition 1.9. Let F be an object of the homotopy monoidal category $(\mathscr{A}, \otimes, \mathbb{1})$, $(I, \leq)$ a finite poset, $\{\lambda_i\}_{i\in I} \subset \mathscr{K}$ a set of scalar objects, $\{\alpha_i : \lambda_i \to F\}_{i\in I}$ a set of morphisms in $\mathscr{A}$, and $\mathbf{P} = (\mathbf{P}_i, d_{ij})_{i,j\in I}$ a twisted complex in $\mathscr{A}$. We say $(\mathbf{P}_i, \alpha_i)_{i\in I}$ is a *diagonalization* of F if $\mathbf{P}$ is an idempotent decomposition of $\mathbb{1} \in \mathscr{A}$, and for all $i \in I$:

$$\mathrm{Cone}(\alpha_i) \otimes \mathbf{P}_i \simeq 0 \simeq \mathbf{P}_i \otimes \mathrm{Cone}(\alpha_i).$$

We would now like to categorify Langrange interpolation so as to use prediaognalizability to construct a diagonalization of $F \in \mathscr{A}$. Letting $c_{ji} = \frac{f-\kappa_j}{\kappa_i-\kappa_j}$, we have that $p_i = \prod_{j\neq i} c_{ji}$. We will then hope to define

$$\mathbf{P}_i = \bigotimes_{j\neq i} \mathbf{C}_{ji},$$

where $\mathbf{C}_{ji}$ categorifies c_{ji}. But how do we categorify c_{ji}? We will make use of the infinite geometric series expansion

$$\begin{aligned} \frac{f-\kappa_j}{\kappa_i-\kappa_j} &= \kappa_i^{-1} \left(\frac{f-\kappa_j}{1-\frac{\kappa_j}{\kappa_i}} \right) \\ &= \kappa_i^{-1}(f-\kappa_j)(1+(\tfrac{\kappa_j}{\kappa_i})+(\tfrac{\kappa_j}{\kappa_i})^2+\cdots) \\ &= \tfrac{1}{\kappa_i}(f-\kappa_j)+\tfrac{\kappa_j}{\kappa_i^2}(f-\kappa_j)+\cdots \end{aligned}$$

We could naively categorify this sum via $\bigoplus_{n\geq 0} \lambda_i^{-1}\,\mathrm{Cone}(\alpha_j) \otimes (\lambda_j\lambda_i^{-1})^{\otimes n}$ given the eigenmap $\alpha_j : \lambda_j \to F$. We will, however, want to also keep track of the eigenmap $\alpha_i : \lambda_i \to F$. This is where the convolution will again save us, allowing for a categorification of sums with more interesting differentials.

Definition 1.10. Let $\alpha : \lambda \to F$ and $\beta : \mu \to F$ be maps from invertible scalar objects λ and μ, such that $\lambda\mu^{-1}$ is small. We then define the complex

$$\mathbf{C}_{\alpha,\beta} = \mathrm{Tot}\left(\begin{array}{ccccccc} & & \frac{\lambda}{\mu} & & & \frac{\lambda^2}{\mu^2} & \\ & \overset{\frac{1}{\mu}\alpha}{\swarrow} & & \overset{-\frac{\lambda}{\mu^2}\beta}{\searrow} \quad \overset{\frac{\lambda}{\mu^2}\alpha}{\swarrow} & & & \overset{-\frac{\lambda^2}{\mu^3}\beta}{\searrow} \\ \frac{1}{\mu}F & & & \frac{\lambda}{\mu^2}F & & & \cdots \end{array} \right)$$

Given the asymmetry of the choice of numerator and denominator, it will be useful to define the similar chain complex below. Note the homological shift $[1]$.

$$\mathbf{C}_{\beta,\alpha}[1] = \mathrm{Tot}\left(\begin{array}{cccccccc} \mathbb{1} & & & \frac{\lambda}{\mu} & & & \frac{\lambda^2}{\mu^2} & \\ & \overset{-\frac{1}{\mu}\beta}{\searrow} \quad \overset{\frac{1}{\mu}\alpha}{\swarrow} & & & \overset{-\frac{\lambda}{\mu^2}\beta}{\searrow} \quad \overset{\frac{\lambda}{\mu^2}\alpha}{\swarrow} & & & \overset{-\frac{\lambda^2}{\mu^3}\beta}{\searrow} \\ & \frac{1}{\mu}F & & & \frac{\lambda}{\mu^2}F & & & \cdots \end{array} \right)$$

Note that if we removed the arrows pointing southwest, the convolution would correspond to our naive guess as to the categorification. We are finally ready to categorify Lagrange interpolation, and hence present our main theorem.

Remark 1.6. In the following theorem, we will use a condition called "strongly commutes" referring to eigenmaps. This is to ensure that eigencones commute, which we will need in our proofs.

Theorem 1.1. *Consider the monoidal homotopy category $(\mathscr{A}, \otimes, \mathbb{1})$ and a categorically prediagonalizable object F with prespectrum $\{\alpha_i : \lambda_i \to F\}_{i\in I}$, where $(I, \leq)$ is a finite totally ordered set, such that each λ_i is invertible and $\lambda_i\lambda_j^{-1}$ is small whenever $j < i$. Then the following categorified projectors constitute an idempotent decomposition of $\mathbb{1}$.*

$$\mathbf{P}_i = \bigotimes_{j\in I} \mathbf{C}_{ji},$$

where $\mathbf{C}_{ji} = \mathbf{C}_{\lambda_j,\lambda_i}$, where we use the first diagram in Definition 1.10 when $j > i$ and the second diagram when $i > j$.

Remark 1.7. To ensure the smallness of $\lambda_i\lambda_j^{-1}$, it is sufficient to suppose that eigenobjects λ_i have distinct homological shifts.

For a proof of this theorem, consult [CITE]. The following section will go through some examples to demonstrate these ideas.

1.3 Preliminary Example

Our first example will take place in a category which serves as a toy model for the category of Soergel bimodules in the simplifying case $n = 2$. This similarity will preempt some of the phenomena we will see in the subsequent section on the full twist.

Example 1.1. Let $A = \mathbb{Z}[C_n]$ where $C_n = \{1, x, ..., x^{n-1}\}$ is the cyclic group of order n, and consider the homotopy category $K^b(A\text{-mod})$ with monoidal structure $\otimes_{\mathbb{Z}}$. Consider the following complex

$$0 \longrightarrow \underline{A} \xrightarrow{1-x} A \xrightarrow{\varepsilon_{x\to 1}} \mathbb{Z} \longrightarrow 0$$

We choose the convention where by writing an element $a \in A$ above an arrow, we indicate the map that multiplies by a, writing $\varepsilon_{x\to 1}$ denotes the map that sets x equal to 1, and underlining a term in the complex sets it to homological degree zero. We will diagonalize the functor $F : K^b(A\text{-mod}) \to K^b(A\text{-mod})$ which acts by tensoring on the left with this complex. Note that, by Morita theory, natural transformations between functors given by tensoring with a complex are specified by chain maps between the corresponding complexes. Suppose we had an eigenmap of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{A} & \longrightarrow & A & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_0}\uparrow & & & & & & \\ & & \mathbb{Z} & & & & & & \end{array}$$

and let's be greedy and expect an eigenobject of the form $M = (0 \to \underline{M} \to 0)$. Then the condition $\mathrm{Cone}(\alpha_0) \otimes_{\mathbb{Z}} M \simeq 0$ requires that

$$0 \longrightarrow M \xrightarrow{\alpha_0(1)} \underline{M[C_n]} \xrightarrow{1-x} M[C_n] \xrightarrow{\varepsilon_{x\to 1}} M \longrightarrow 0$$

has zero homology and thus $\mathrm{Ker}(1-x) = \mathrm{Im}\,\alpha_0(1)$, forcing α_0 to be the map

$$\alpha_0 : 1 \mapsto 1 + x + x^2 + ... + x^{n-1}.$$

The reader can check in this case that for $M = A$, the complex $\mathrm{Cone}(\alpha_0) \otimes A$ is indeed homotopic to 0, and thus A is a weak eigenobject of F with eigenmap α_0. An eigenmap of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{A} & \longrightarrow & A & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & & & {\scriptstyle\alpha}\uparrow & & & & \\ & & & & \mathbb{Z} & & & & \end{array}$$

is not possible since there are no such nonzero chain maps. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{A} & \longrightarrow & A & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle \alpha_1} & & \\ & & & & & & \mathbb{Z} & & \end{array}$$

were an eigenmap with eigenobject M. Then

$$0 \longrightarrow \underline{M[C_n]} \xrightarrow{(1-x,0)} M[C_n] \oplus M \xrightarrow{\varepsilon_{x\to 1} \oplus \alpha_1(1)} M \longrightarrow 0$$

must have zero homology. However, there is no way to choose α_1 so that there is no homology in degree 1. Thus we are forced to consider a more complicated eigenobject (one that is not concentrated in a single homological degree). Luckily, since there are only two possible eigenmaps, if we are to succeed we must have

$$\mathrm{Cone}(\alpha_1) \otimes_{\mathbb{Z}} \mathrm{Cone}(\alpha_0) \simeq 0.$$

Thus $\mathrm{Cone}(\alpha_0)$ must be the eigenobject corresponding to α_1. With the choices $\alpha_0 = (1+x+\ldots+x^{n-1})$, $M = A$, and $\alpha_1 = 1$ we will argue that this is indeed the case. $\mathrm{Cone}(\alpha_0)$ now looks like

$$0 \longrightarrow \underline{A} \xrightarrow{(1-x,0)} A \oplus \mathbb{Z} \xrightarrow{\varepsilon_{x\to 1} \oplus 1} \mathbb{Z} \longrightarrow 0$$

which via Gaussian elimination is seen to be homotopic to

$$0 \longrightarrow \underline{A} \xrightarrow{1-x} A \longrightarrow 0 \longrightarrow 0$$

and so, since we already have $\mathrm{Cone}(\alpha_1) \otimes A \simeq 0$,

$$\mathrm{Cone}(\alpha_1) \otimes \mathrm{Cone}(\alpha_0) \simeq \mathrm{Cone}\left(\mathrm{Cone}(\alpha_1) \otimes A \xrightarrow{1 \otimes (1-x)} \mathrm{Cone}(\alpha_1) \otimes A \right) \simeq 0.$$

Thus we have shown that F is prediagonalizable. We now use α_0 and α_1 to construct the categorified projectors. $\mathbf{P}_0$ is the total complex of the diagram in Figure 1.1. Applying Gaussian elimination to this total complex we find that

$$\mathbf{P}_0 \simeq \cdots \xrightarrow{\left(\frac{x^n-1}{1-x}\right) \circ \varepsilon_{x\to 1}} \mathbb{Z}[C_n] \xrightarrow{1-x} \mathbb{Z}[C_n] \xrightarrow{\left(\frac{x^n-1}{1-x}\right) \circ \varepsilon_{x\to 1}} \mathbb{Z}[C_n] \xrightarrow{1-x} \mathbb{Z}[C_n] \longrightarrow \underline{0}.$$

Similarly, $\mathbf{P}_1$ is the total complex of the diagram shown in Figure 1.2. After applying Gaussian elimination, we get

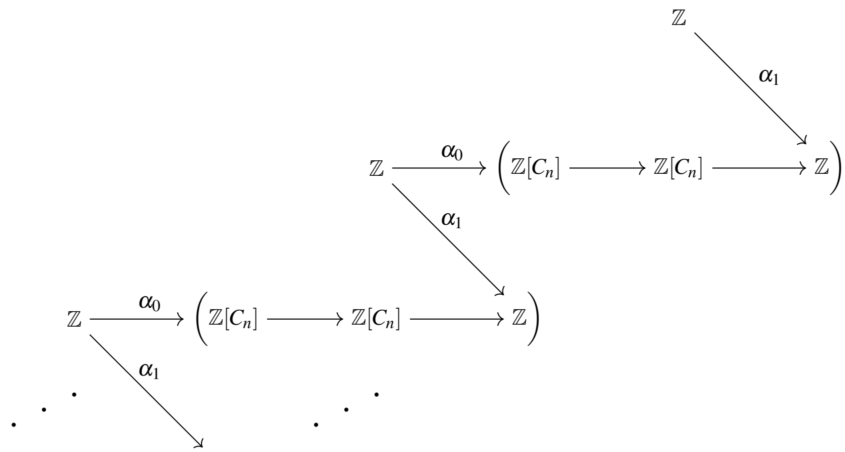


Fig. 1.1 To get the projector $\mathbf{P}_0$ we take the total complex of this diagram

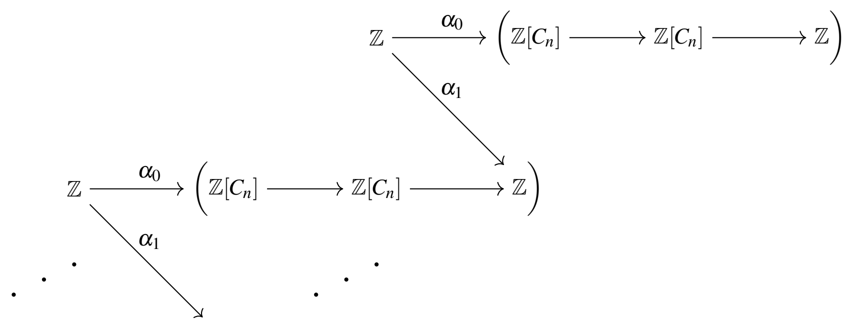


Fig. 1.2 To get $\mathbf{P}_1$ we take a total complex of this diagram

$$\mathbf{P}_1 \simeq \cdots \xrightarrow{\left(\frac{x^n-1}{1-x}\right)\circ \varepsilon_{x\to 1}} \mathbb{Z}[C_n] \xrightarrow{1-x} \mathbb{Z}[C_n] \xrightarrow{\left(\frac{x^n-1}{1-x}\right)\circ \varepsilon_{x\to 1}} \mathbb{Z}[C_n] \xrightarrow{1-x} \mathbb{Z}[C_n] \xrightarrow{\varepsilon_{x\to 1}} \underline{\mathbb{Z}}.$$

1.4 Diagonalizing the Full Twist

1.4.1 Jucys-Murphy Elements

Here we will provide some background information for our flagship example categorical diagonalization in the context of Soergel bimodules. We will first define Jucys-Murphy elements. Recall the map Z from the braid group $\mathrm{Br}(S_n)$ to the Hecke algebra $\mathbb{H}(S_n)$ which sends σ_i to H_{s_i}.

Definition 1.11. The *Jucys-Murphy element* $j_i \in \mathbb{H}(S_n)$ is defined as

$$j_i = Z(\sigma_{i-1}...\sigma_2\sigma_1\sigma_2...\sigma_{i-1}),$$

Diagramatically the Jucys-Murphy elements can be defined as the image of the pure braid indicated in Figure 1.3. Note that these elements have an interesting commu-

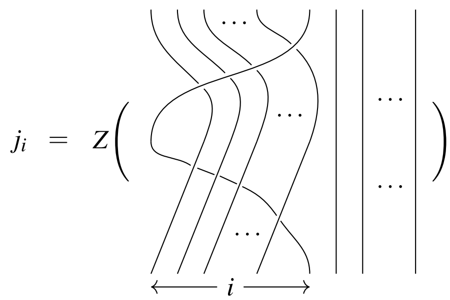


Fig. 1.3 The Jucys-Murphy element j_i

tator property: $[j_i, j_k] = 0$ for $k < i$. Furthermore, these elements play an important role in the theory of irreducible representations of $\mathbb{H}(S_n)$.

Theorem 1.2. *There is a one-to-one correspondence between irreducible representations of $\mathbb{H}(S_n)$ and partitions λ of n. Moreover, the irreducible representation V_λ corresponding to the partition λ has a Young basis indexed by the* standard Young tableaux $\{e_T\}_{T\in SYT(\lambda)}$*, which also serves as a simultaneous eigenbasis for $j_1, j_2, ..., j_n$.*

Remark 1.8. The standard Young tableaux $\mathrm{SYT}(\lambda)$ of shape λ are fillings of the Young diagram λ with the numbers $1,\dots,n$ in such a way that the numbers in the rows and columns are increasing rightward and downward.

Given a Young tableau, let $x(\boxed{k})$ denote content of the box containing the number k (content was defined in the previous lecture). Then one can check that j_k has the following action on e_T:

$$j_k e_T = v^{2x(\boxed{k})} e_T.$$

We now define the operator we will be interested in diagonalizing.

Definition 1.12. The *full twist on n strands* is defined as

$$\mathrm{ft}_n = H_{w_0}^2 = j_1 j_2 ... j_n.$$

Recall from a previous lecture that $x(\lambda)$ denotes total content of λ, and $c(\lambda)$ denotes the total column number of λ, ft_n acts on V_λ by $v^{2x(\lambda)}$, and by the theorem of Graham, Mathas, Lusztig, we have:

$$H_{w_0} b_{(P,Q,\lambda)} = (-1)^{c(\lambda)} v^{x(\lambda)} b_{(P^\vee,Q,\lambda)} + \text{ lower terms.}$$

Thus, since $(P^\vee)^\vee = P$ we find that

$$\mathrm{ft}_n b_{(P,Q,\lambda)} = (-1)^{2c(\lambda)} v^{2x(\lambda)} b_{(P,Q,\lambda)} + \text{ lower terms}$$

so $b_{(P,Q,\lambda)}$ is an eigenvector of ft_n modulo lower terms. A corollary of this is the following:

Theorem 1.3. *The left cell representation $V_{(-,Q,\lambda)}$ is isomorphic to V_λ.*

1.4.2 Diagonalizing Rouqier Complexes

We want to categorify everything in the previous section, in particular the eigenvectors of the full twist. We have seen that the Rouqier complex F_s categorifies the Hecke algebra element H_s, and tensor products of Rouqier complexes categorify products of H_{s_i}'s. For example, $\mathrm{FT}_2 = F_s F_s$ categorifies $\mathrm{ft}_2 = H_s H_s$. We will focus on the case $n = 2$ and diagonalize the full twist FT_2. Consider $H_s = H_{s_1} \in \mathbb{H}(S_n)$. We know from the definition of a Hecke algebra that

$$H_s^2 = (v^{-1} - v)H_s + 1$$

and thus

$$(H_s - v^{-1})(H_s + v) = 0.$$

Therefore, H_s has eigenvalues v^{-1} and $-v$, from which we conclude that $\mathrm{ft}_2 = H_s^2$ has eigenvalues v^2 and v^{-2}.

We begin the discussion of the categorification of this picture by recalling an earlier computation:

$$\mathrm{FT}_2 \simeq \cdots \longrightarrow 0 \longrightarrow \underline{B_s(-1)} \xrightarrow{\updownarrow|-|\updownarrow} B_s(1) \xrightarrow{\uparrow} R(2)$$

Tensoring with $\underline{B_s}$ gives

$$\mathrm{FT}_2 \otimes \underline{B_s} \; \simeq \; \cdots \longrightarrow 0 \longrightarrow \underline{B_s B_s(-1)} \xrightarrow{\updownarrow||-|\updownarrow|} B_s B_s(1) \xrightarrow{\uparrow|} B_s(2)$$

where, in general, $\underline{M}$ will denote the chain complex $0 \to \underline{M} \to 0$. When context is clear, we will henceforth make the identification $\underline{M} = M$. Also, recall that $B_s B_s(-1) = B_s(0) \oplus B_s(-2)$ and $B_s B_s(1) = B_s(2) \oplus B_s(0)$. Using this fact, we can apply Gaussian elimination to the above complex to get

$$\mathrm{FT}_2 \otimes B_s \simeq B_s(-2) = \mathbb{1}(-2)[0] \otimes B_s$$

We hence say that B_s is a weak categorical eigenobject with eigenvalue $\mathbb{1}(-2)[0]$, a shift of the monoidal unit $\underline{R}$.

Using the previous example as motivation we construct eigenmaps

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_s(-1) & \longrightarrow & B_s(1) & \longrightarrow & R(-2) & \longrightarrow & 0 \\
 & & \Big\uparrow {\scriptstyle \downarrow\!\bullet \;\; \alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}} & & & & & & \\
 & & R(-2) & & & & & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_s(-1) & \longrightarrow & B_s(1) & \longrightarrow & R(-2) & \longrightarrow & 0 \\
 & & & & & & \Big\uparrow {\scriptstyle 1 \;\; \alpha_{\square\square}} & & \\
 & & & & & & R(-2) & &
\end{array}
$$

The reader can check that $\operatorname{Cone}\left(\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) \otimes B_s \simeq 0$ so B_s is an eigenobject for FT_2 corresponding to the eigenmap $\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ with eigenvalue $\mathbb{1}(-2)[0]$ (as computed in the previous paragraph). The computation

$$\operatorname{Cone}\left(\alpha_{\square\square}\right) \otimes \operatorname{Cone}\left(\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) \simeq 0$$

tells us that $\operatorname{Cone}\left(\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right)$ is an eigenobject for FT_2 corresponding to the eigenmap $\alpha_{\square\square}$ with eigenvalue $\mathbb{1}(2)[-2]$. Note that the above homotopy decategorifies to the prediagonalizability condition

$$(\mathrm{ft}_2 - v^{-2})(\mathrm{ft}_2 - v^2) = 0.$$

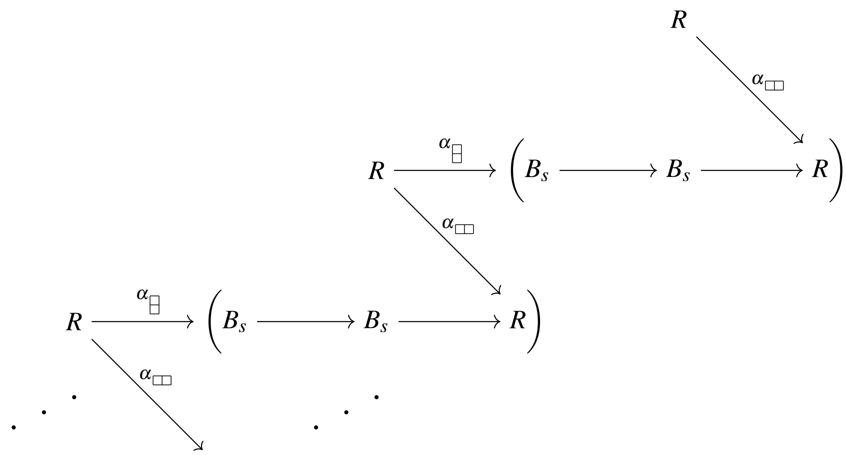
Now let us compute the projectors. At the decategorified level, the eigenvalue v^{-2} of ft_2 corresponding to the partition $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ gives rise (via Lagrange interpolation) to the projector

$$p_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{\mathrm{ft}_2 - v^2}{v^{-2} - v^2} = \frac{b_s}{v + v^{-1}},$$

and the eigenvalue v^2 corresponding to the partition $\square\square$ gives rise to

$$p_{\square\square} = \frac{\mathrm{ft}_2 - v^{-2}}{v^2 - v^{-2}} = 1 - \frac{b_s}{v + v^{-1}}.$$

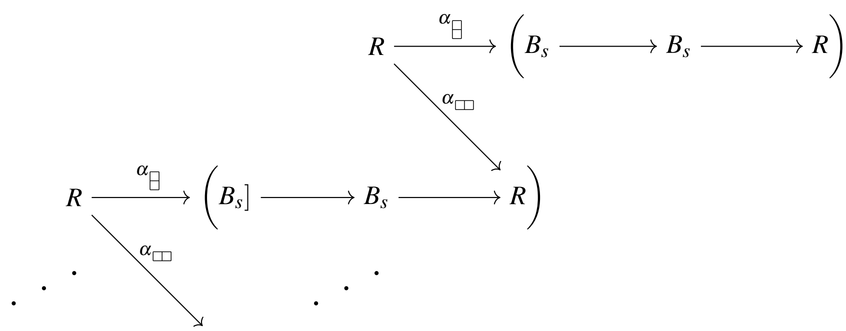
Our goal is to diagonalize the full twist $\mathrm{FT}_2 = F_s F_s$. We use the eigenmaps $\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $\alpha_{\square\square}$ to construct the categorified projectors $P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $P_{\square\square}$. To get $P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ we take the total complex of the following diagram:



Applying Gaussian elimination to this complex, we find that

$$P_{\boxminus} \simeq \cdots \longrightarrow B_s \longrightarrow B_s \longrightarrow B_s \longrightarrow \underline{B_s} \longrightarrow 0.$$

Similarly, $P_{\square\square}$ is the total complex of the diagram shown below:



Applying Gaussian elimination yields:

$$P_{\square\square} \simeq \cdots \longrightarrow B_s \longrightarrow B_s \longrightarrow B_s \longrightarrow B_s \longrightarrow \underline{R}.$$

The homotopy

$$\mathrm{Cone}\left(P_{\square\square}[-1] \to P_{\boxminus}\right) \simeq \mathbb{1}$$

decategorifies to the idempotent decomposition

$$p_{\square\square} + p_{\boxminus} = 1.$$

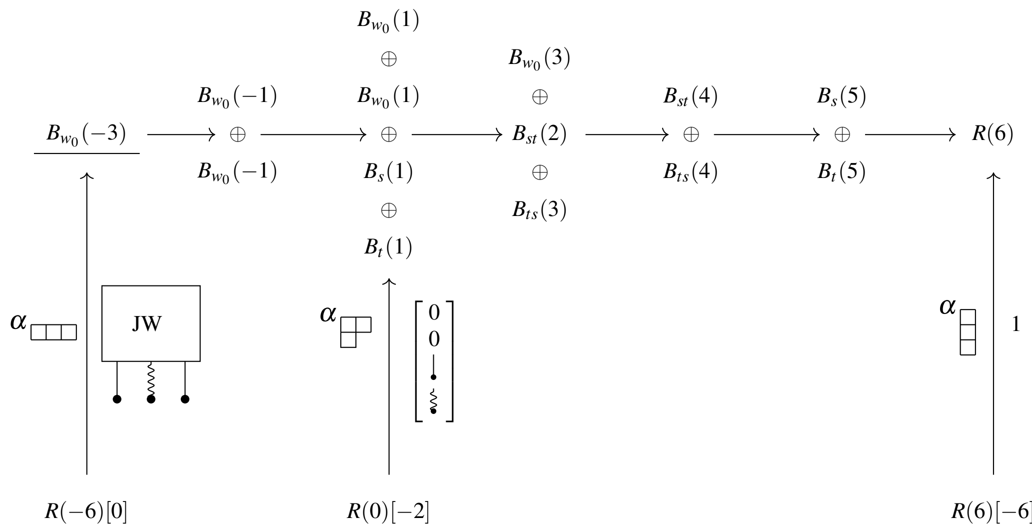


Fig. 1.4 The picture for FT_3.

For the case of FT_3 we get the picture shown in Figure 1.4. An ambitious reader might try to work out what the projectors look like in this case.

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