Torsion in Khovanov Homology of 3-braids

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Abstract

In the Khovanov homology of link, the presence of odd torsion is rare and some families of links are conjectured to have only $\mathbb{Z}_2$-torsion. We give a partial answer to Sazdanović and Przytycki's conjecture that 3-braids have only $\mathbb{Z}_2$-torsion by using spectral sequences on Khovanov homology to exhibit an infinite family of 3-braids, containing all 3-strand torus links, which has only $\mathbb{Z}_2$-torsion in Khovanov homology. We give an explicit computation of integral Khovanov homology for all links in this family.

Contents

1 Introduction **1**

2 Khovanov Homology **4**
2.1 Construction of Khovanov Homology 4
2.2 Computational Tools 5

3 Excluding Odd Torsion **6**
3.1 Reducing to the Case $n \geq 0$ 6
3.2 Odd Torsion in $\Omega_0, \Omega_1, \Omega_2$ 7
3.3 Odd Torsion in Ω_3 7

4 Spectral Sequences and Even Torsion **8**
4.1 The Bockstein Spectral Sequence 9
4.2 Bar-Natan Homology and The Turner Spectral Sequence 10
4.3 Even Torsion in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ 12

1 Introduction

In 1984, V. Jones discovered the Jones polynomial [4], a powerful link invariant. In 1999, M. Khovanov [5] categorified the Jones polynomial $J(L)$ to a bigraded homology theory $H(L)$, which is a stronger link invariant, called the Khovanov homology. Khovanov homology categorifies the Jones polynomial in the sense that the Jones polynomial of a link can be recovered as the graded Euler characteristic of its Khovanov homology. Each $H^{i,j}(L)$ is a finitely generated abelian group, and the Jones polynomial gets a contribution only from the free part of $H(L)$. Thus torsion in Khovanov homology is a new phenomena in knot theory which does not appear in the theory of Jones polynomials. Throughout this paper we will use the term '$\mathbb{Z}_{p^r}$-torsion', p a prime, to mean a direct summand of isomorphism class $\mathbb{Z}_{p^r}$ in the primary decomposition of the integral Khovanov homology of a link. Since the genesis of Khovanov homology, several authors have studied its torsion. In [12], Shumakovitch shows that non-split alternating links have only $\mathbb{Z}_{2^r}$-torsion, and in [13] he uses a relationship between the Turner and Bockstein differentials on $\mathbb{Z}_2$-Khovanov homology to show in fact there is only $\mathbb{Z}_2$-torsion. In [3], Przytycki, Helme-Guizon, and Rong establish a connection between the Khovanov homology of a link and the chromatic graph homology of certain graphs associated to diagrams of

the link. In [7], Lowrance and Sazdanovic use this connection to show that in a certain range of homological gradings, Khovanov homology contains only $\mathbb{Z}_2$-torsion. In [11], Przytycki and Sazdanovic obtain explicit formulae for torsion and prove that Khovanov homology of semi-adequate links contains $\mathbb{Z}_2$-torsion if the corresponding Tait-type graph has a cycle of length at least 3. In [11], Przytycki and Sazdanovic conjecture the following, connecting torsion in Khovanov homology to braid index:

Conjecture 1.1 (PS braid conjecture, 2012)**.**

1. *Khovanov homology of a closed 3-braid can have only $\mathbb{Z}_2$-torsion.*
2. *Khovanov homology of a closed 4-braid cannot have $\mathbb{Z}_{p^r}$-torsion for $p \neq 2$.*
3. *Khovanov homology of a closed 4-braid can have only $\mathbb{Z}_2$ and $\mathbb{Z}_4$-torsion.*
4. *Khovanov homology of a closed n-braid cannot have $\mathbb{Z}_{p^r}$-torsion for $p > n$ (p prime).*
5. *Khovanov homology of a closed n-braid cannot have $\mathbb{Z}_{p^r}$-torsion for $p^r > n$.*

Counterexamples to parts 2,3 and 5 are given in [9] but there are no known counterexamples to parts 4, while computations and partial results suggest that part 1 is indeed true. The goal of this (ongoing) project is to prove that closed 3-braids have only $\mathbb{Z}_2$-torsion in Khovanov homology.

For link diagrams, will use the conventions for positive/negative crossings and smoothings shown in Figure 1 for crossings and smoothings in link diagrams. Consider the braid group B_s on s-strands whose generators

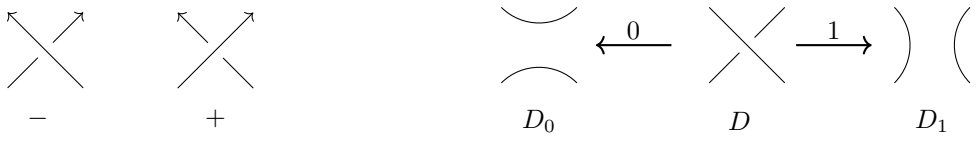


Figure 1: Conventions for crossings and smoothings in link diagrams.

are shown in Figure 2. The *half twist* $\Delta \in B_s$ is defined as $\Delta = (\sigma_1\sigma_2 \dots \sigma_{p-1})(\sigma_1\sigma_2 \dots \sigma_{p-2}) \dots (\sigma_1\sigma_2)(\sigma_1)$,

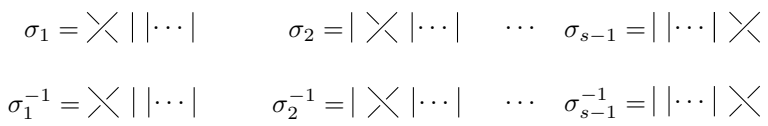


Figure 2: Generators and their inverses for the braid group B_s.

and the *full twist* is Δ^2. The closure of a braid diagram is a diagram of a link, and a famous result of Alexander tells us that every link can be represented by a diagram arising as the closure of a braid. For convenience, throughout this paper, a braid word will be used to refer to either an element of the braid group or its braid closure depending on the context in which it appears. If two elements of the braid group are conjugate, then the closure of two braids are isotopic. For purposes of computation, one may desire a classification of elements of the braid group B_n up to conjugacy (conjugate braid words represent isotopic links). For $s = 2$ of course the classification is trivial. For $s \geq 4$, no classification is known. For $s = 3$, Murasugi [10] provides us with the following:

Theorem 1.2. *Every element of the braid group B_3 is conjugate to a unique element of one of the following disjoint sets:*

$$\Omega_0 = \{\Delta^{2n} \mid n \in \mathbb{Z}\}$$
$$\Omega_1 = \{\Delta^{2n}\sigma_1\sigma_2 \mid n \in \mathbb{Z}\}$$

$$\Omega_2 = \{\Delta^{2n}(\sigma_1\sigma_2)^2 \mid n \in \mathbb{Z}\}$$
$$\Omega_3 = \{\Delta^{2n+1} \mid n \in \mathbb{Z}\}$$
$$\Omega_4 = \{\Delta^{2n}\sigma_1^{-p} \mid n \in \mathbb{Z}\}$$
$$\Omega_5 = \{\Delta^{2n}\sigma_2^{q} \mid n \in \mathbb{Z}\}$$
$$\Omega_6 = \{\Delta^{2n}\sigma_1^{-p_1}\sigma_2^{q_1}\dots\sigma_1^{-p_r}\sigma_2^{q_r} \mid n \in \mathbb{Z}\}$$

where p, q, p_i and q_i are positive integers.

In this paper we prove the following partial result, in an effort toward proving part 1 of the PS braid conjecture.

Theorem 4.7. *3-braids of type $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ have only $\mathbb{Z}_2$-torsion in Khovanov homology.*

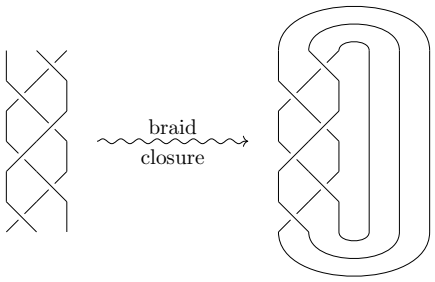
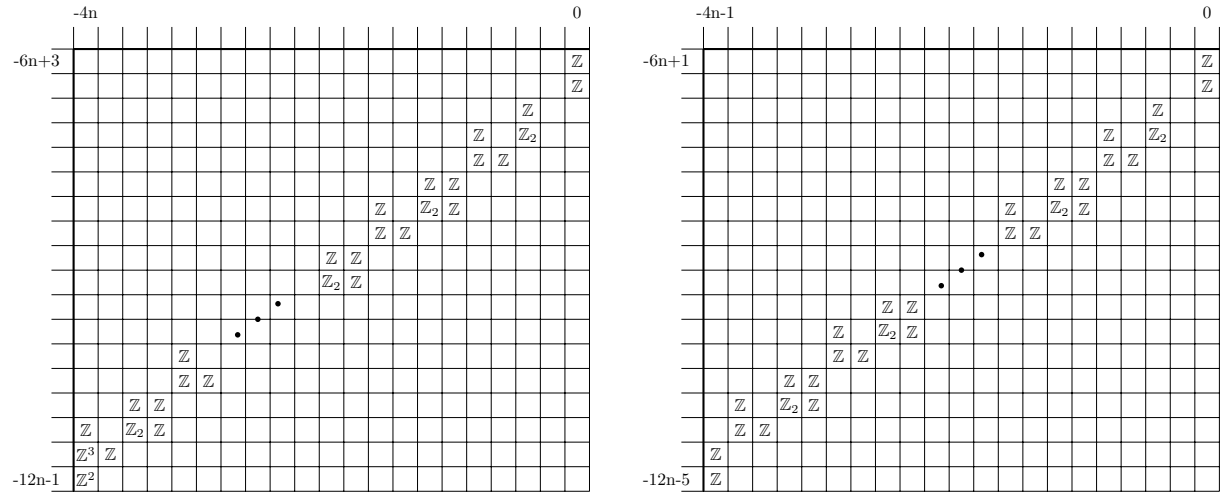


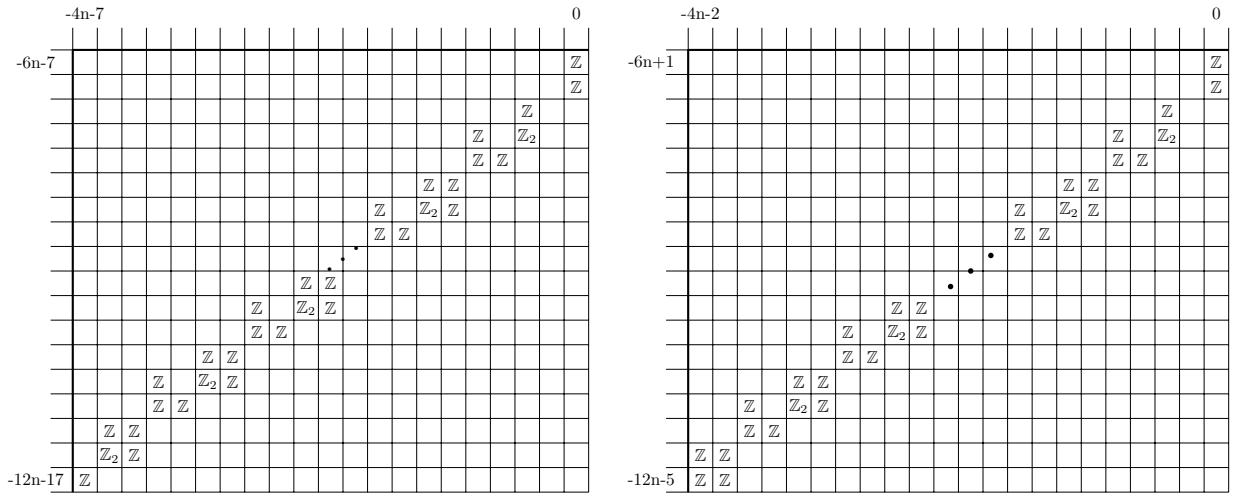
Figure 3: A braid diagram and its braid closure.

We also give explicit computations of the integral Khovanov homology for links in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$, shown in Figures 4a, 4b, 5a, and 5b. To prove Theorem 1 we use Benheddi's computation [2] of $\mathbb{Z}_2$ Khovanov homology of 3-strand torus links together with Shumakovitch's result [13] using Khovanov's vertical differential to compare the Turner spectral sequence to the $\mathbb{Z}_2$-Bockstein spectral sequence on $\mathbb{Z}_2$-Khovanov homology.



(a) (b)

Figure 4: In (a) we have the Integral Khovanov homology of $T(3, 3n)$ over $\mathbb{Z}$. In (b) we have the Integral Khovanov homology of $T(3, 3n+1)$ over $\mathbb{Z}$.



(a) (b)

Figure 5: In (a) we have the Integral Khovanov homology of $T(3, 3n+2)$ over $\mathbb{Z}$. In (b) we have Integral Khovanov homology of the braid closure of Δ^{2n+1} over $\mathbb{Z}$.

This paper will be organized as follows. In Section 2, we review the definition of Khovanov homology of links and give computational tools in the form of long exact sequences. In Section 3 we explain why links in the classes $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ have no odd torsion in Khovanov homology. Finally, in Section 4, we review the relevant spectral sequences in Khovanov homology and apply them to conclude links in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ have only $\mathbb{Z}_2$ torsion in Khovanov homology.

2 Khovanov Homology

Khovanov homology is an invariant of oriented links $L \subset S^3$ with values in the category of bigraded modules over a ring R with identity. We begin with a construction of Khovanov homology.

2.1 Construction of Khovanov Homology

Let R be a ring with identity. A bigraded R-module is an R-module M with a direct sum decomposition of the form $M = \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}$. The submodule $M^{i,j}$ is said to have *bigrading* (i, j). For our purposes we will refer to i as the *homoligical grading* and to j as the *polynomial grading*. Given two bigraded R-modules $M = \bigoplus_{i,j\in\mathbb{Z}} M^{i,j}$ and $N = \bigoplus_{i,j\in\mathbb{Z}} N^{i,j}$, we define the direct sum $M \oplus_R N$ and tensor product $M \otimes_R N$ to be the bigraded R-modules with components $(M \oplus N)^{i,j} = M^{i,j} \oplus N^{i,j}$ and $(M \otimes N)^{i,j} = \bigoplus_{k+m=i,l+n=j} M^{k,l} \otimes N^{m,n}$. We also define homological and polynomial shift operators, denoted $[\cdot]$ and $\{\cdot\}$, respectively, by $(M[r])^{i,j} = M^{i-r,j}$ and $(M\{s\})^{i,j} = M^{i,j-s}$.

Consider the directed graph whose vertex set $\mathcal{V}(n) = \{0, 1\}^n$ comprises n-tuples of 0's and 1's and whose edge set $\mathcal{E}(n)$ contains a directed edge from $I \in \mathcal{V}(n)$ to $J \in \mathcal{V}(n)$ if and only if all entries of I and J are equal except for one, where I is 0 and J is 1. One can think of the underlying graph as the 1-skeleton of the n-dimensional cube. For example, if $n = 4$ there are two outward edges starting from $(0, 1, 0, 1)$, one ending at $(0, 1, 1, 1)$ and the other ending at $(1, 1, 0, 1)$. The height of a vertex $I = (k_1, k_2, \ldots, k_n)$ is $h(I) = k_1 + k_2 + \cdots + k_n$. In other words, $h(I)$ is the number of 1's in I. If ε is an edge from I to J and they differ in the r-th entry, the height of ε is $|\varepsilon| := h(I)$. The sign of ε is $(-1)^{\varepsilon} := (-1)^{\sum_{i=1}^{r-1} k_i}$. In other words, the sign of ε is -1 if the number of 1's before the r-th entry is odd, and is $+1$ if even.

Let D be a diagram of a link L with n crossings $c_1, c_2, \ldots, c_n$. To each vertex $I \in \mathcal{V}(n)$ we associate a collection of circles $D(I)$, called a Kauffman state, obtained by 0-resolving those crossings c_i for which $k_i = 0$ and 1-resolving those crossings c_j for which $k_j = 1$. The 0- and 1-resolution conventions are given in Figure 1. To each Kauffman state we associate a bigraded R-module $C(D(I))$ as follows. Let $\mathcal{A}_2 = R[X]/X^2 = R1 \oplus RX$ be the bigraded module with generator 1 in bigrading $(0,1)$ and generator X in bigrading $(0,-1)$. Denoting the number of circles in $D(I)$ by $|D(I)|$, we define

$$C(D(I)) := \mathcal{A}_2^{\otimes |D(I)|}[h(I)]\{h(I)\}.$$

Here, each tensor factor of $\mathcal{A}_2$ is understood to be associated to a particular circle of $D(I)$. Having associated modules to vertices, we now associated maps to directed edges. Define R-module homomorphisms $m : \mathcal{A}_2 \otimes \mathcal{A}_2 \to \mathcal{A}_2$ and $\Delta : \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$ (called multiplication and comultiplication, respectively) by

$$m(1 \otimes 1) = 1 \quad m(1 \otimes X) = X \quad m(X \otimes 1) = X \quad m(X \otimes X) = 0$$

$$\Delta(1) = 1 \otimes X + X \otimes 1 \quad \Delta(X) = X \otimes X$$

To an edge ε from I to J we associate the map $d_\varepsilon : C(D(I)) \to C(D(J))$ defined as follows.

(1) If $|D(J)| = |D(I)| - 1$, then $D(J)$ is obtained by merging two circles of $D(I)$ into one: d_ε acts as multiplication m on the tensor factors associated to the circles being merged, and acts as the identity map on the remaining tensor factors.

(2) If $|D(J)| = |D(I)| + 1$ then $D(J)$ is obtained by splitting one circle of $D(I)$ into two: d_ε acts as comultiplication Δ on the tensor factor associated to the circle being split, and acts as the identity map on the remaining tensor factors.

Suppose D has n_+ positive crossings and n_- negative crossings. Define the bigraded module

$$C^{*,*}(D) = \bigoplus_{I \in \mathcal{V}(n)} C(D(I))[-n_-]\{n_+ - 2n_-\}. \tag{1}$$

Define maps $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$ by $d^i = \sum_{|\varepsilon|=i} (-1)^\varepsilon d_\varepsilon$. In [5], Khovanov shows that $(C(D), d)$ is a (co)chain complex, and since the maps d_i are polynomial degree-preserving, we get a bigraded homology R-module

$$H(L, R) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L, R)$$

called the Khovanov homology of the link L with coefficients in R. Proofs that d is a differential and that these homology groups are independent of the diagram D and the ordering of the crossings can be found in [5, 16, 1]. If $R = \mathbb{Z}$ we simply write $H(L) = H(L, \mathbb{Z})$. In this article we will focus on the rings $R = \mathbb{Z}, \mathbb{Q}$ and $\mathbb{Z}_p$, where p is a prime.

The (unnormalized) Jones polynomial is recovered as the graded Euler characteristic of Khovanov homology:

$$\hat{J}_L(q) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}}(H^{i,j}(L) \otimes \mathbb{Q})$$

2.2 Computational Tools

Recall, given an oriented link diagram D there is a long exact sequence relating D to the zero and one smoothings D_0 and D_1 at a given crossing of D. If the crossing is negative, we set $c = n_-(D_0) - n_-(D)$ to be the number of negative crossings in D_0 minus the number in D. For each j there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(D_1) \longrightarrow H^{i,j}(D) \longrightarrow H^{i-c,j-3c-1}(D_0) \xrightarrow{\delta_*} H^{i+1,j+1}(D_1) \longrightarrow \tag{2}$$

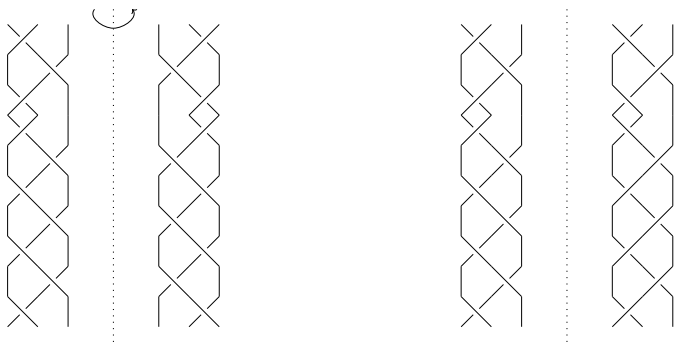


Figure 6: Left: The braid word $\omega = \Delta^2\sigma_1^2\sigma_2\sigma_1 \in \Omega_6$ and the corresponding diagram $\phi(\omega) = \Delta^2\sigma_2^2\sigma_1\sigma_2$. Think of $\phi(D)$ as D rotated about the dotted line. Right: The braid word ω and its mirror image $m(\omega)$.

Note that in this case D_1 inherits an orientation, and D_0 must be given one. Similarly, if the crossing is positive, we set $c = n_-(D_1) - n_-(D)$ to be the number of negative crossings in D_1 minus the number in D. For each j there is a long exact sequence

$$\xrightarrow{\delta_*} H^{i-c-1,j-3c-2}(D_1) \longrightarrow H^{i,j}(D) \longrightarrow H^{i,j-1}(D_0) \xrightarrow{\delta_*} H^{i-c,j-3c-2}(D_1) \longrightarrow \tag{3}$$

and note that in this case D_0 inherits an orientation, and D_1 must be given one.

3 Excluding Odd Torsion

3.1 Reducing to the Case $n \geq 0$

In the following sections we will demonstrate that 3-braids have no $\mathbb{Z}_{p^r}$-torsion for $p \neq 2$ in Khovanov homology, in the case that $n \geq 0$. The cases $n < 0$ follow from the following observations.

1. The mirror image $m(D)$ of a knot diagram D is the diagram gotten by changing all crossings. On the level of braid words, $m : B_3 \to B_3$ is a group homomorphism satisfying $m(\sigma_i) = \sigma_i^{-1}$ and $m(\Delta) = \Delta^{-1}$. Recall that the torsion in Khovanov homology of a link diagram and the torsion of its mirror image differ only by a homological shift [5, Corollary 11], so L has $\mathbb{Z}_{p^r}$-torsion if and only if $m(L)$ has $\mathbb{Z}_{p^r}$-torsion.

2. Consider the group homomorphism $\phi : B_3 \to B_3$ defined on generators by $\phi(\sigma_1) = \sigma_2$ and $\phi(\sigma_2) = \sigma_1$. If the braid word ω is a projection of a link L embedded in $\{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1\}$ to the plane $z = 0$, then the projection of L to the plane $z = 1$ is $\phi(\omega)$. Thus the map ϕ preserves the isotopy type of the braid word. Therefore ω has $\mathbb{Z}_{p^r}$-torsion in Khovanov homology if and only if $\phi(\omega)$ has $\mathbb{Z}_{p^r}$-torsion. Note that the homomorphism ϕ satisfies $\phi(\Delta) = \Delta$. See Figure 6 for an example of the action of ϕ on a braid diagram.

The following equalities together with the above two arguments show that in all cases $\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_4', \Omega_5', \Omega_6'$

it suffices to determine torsion for $n \geq 0$:

$$m(\Delta^{-2n}) = \Delta^{2n} \tag{4}$$
$$m(\Delta^{-2n}\sigma_1\sigma_2) = (\sigma_1\sigma_2)^{3n-1} \tag{5}$$
$$m(\Delta^{-2n}(\sigma_1\sigma_2)^2) = (\sigma_1\sigma_2)^{3n-2} \tag{6}$$
$$m(\Delta^{-2n-1}) = \Delta^{2n+1} \tag{7}$$
$$m\phi(\Delta^{-2n}\sigma_1^{-p}) = \Delta^{2n}\sigma_2^{p} \tag{8}$$
$$m\phi(\Delta^{-2n}\sigma_2^{q}) = \Delta^{2n}\sigma_1^{-q} \tag{9}$$
$$m\phi(\Delta^{-2n}\sigma_1^{-p_1}\sigma_2^{q_1}\ldots\sigma_1^{-p_r}\sigma_2^{q_r}) = \Delta^{2n}\sigma_2^{p_1}\sigma_1^{-q_1}\ldots\sigma_2^{p_r}\sigma_1^{-q_r}. \tag{10}$$

3.2 Odd Torsion in $\Omega_0, \Omega_1, \Omega_2$

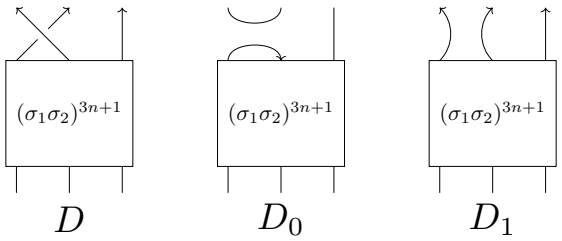
We begin with a simple fact (shown by Turner in [14]) that will be useful in conjunction with the classification of 3-braids in Section 1 and the long exact sequence from Section 2.2.

Lemma 3.1. *For each $q \in \mathbb{Z}$, the Khovanov homology $H(T(3,q))$ of the torus link $T(3,q)$ contains no $\mathbb{Z}_{p^r}$-torsion for $p \neq 2$. That is, there is no $\mathbb{Z}_{p^r}$-torsion for in $p \neq 2$ in the Khovanov homology of links of types Ω_0, Ω_1 and Ω_2.*

Proof. Actually, the result in [14] tells us that for $p \neq 2$ $T(3,q)$ has no $\mathbb{Z}_{p^r}$-torsion in Khovanov homology for $q \geq 2$. $T(3,0)$ and $T(3,1)$ are both the unlink so the result is true for $q \geq 0$. To get negative values of q, simply use (4), (5), (6) above and the fact that $\Delta^{2n} = T(3,3n)$, $\Delta^{2n}\sigma_1\sigma_2 = T(3,3n+1)$ and $\Delta^{2n}(\sigma_1\sigma_2)^2 = T(3,3n+2)$. □

3.3 Odd Torsion in Ω_3

First observe that $\Delta^{2n+1} = (\sigma_1\sigma_2)^{3n+1}\sigma_1$ where $(\sigma_1\sigma_2)^{3n+1}$ is a braid word for $T(3,3n+1)$. We consider smoothing the top σ_1.



Notice that D_0 is a diagram for the unknot U and D_1 is a diagram for $T(3,3n+1)$. The top σ_1 in Δ^{2n+1} is a negative crossing so we compute $c = n_-(D_0) - n_-(D) = (1+2n) - (6n+3) = -4n-2$. Using (2) for each j, and letting $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ where p is an odd prime, we get a long exact sequence

$$\xrightarrow{\delta_*} H^{i,j+1}(T(3,3n+1),\mathbb{F}) \longrightarrow H^{i,j}(D,\mathbb{F}) \longrightarrow H^{i+4n+2,j+12n+5}(U,\mathbb{F}) \xrightarrow{\delta_*} H^{i+1,j+1}(T(3,3n+1),\mathbb{F}) \tag{11}$$

For $i \neq -4n-2, -4n-1$, we have $H^{i+4n+2,j+12n+5}(U,\mathbb{F}) = 0 = H^{i+4n+1,j+12n+5}(U,\mathbb{F})$ for every j, so exactness yields $H^{i,j}(D,\mathbb{F}) \cong H^{i,j+1}(T(3,3n+1),\mathbb{F})$ for every j. For $j \neq -12n-5\pm 1$, the portion of the long exact sequence containing $i = -4n-2$ and $-4n-1$ looks like

$$0 \longrightarrow H^{-4n-2,j+1}(T(3,3n+1),\mathbb{F}) \longrightarrow H^{-4n-2,j}(D,\mathbb{F}) \longrightarrow 0$$

$$\xrightarrow{\delta_*} H^{-4n-1,j+1}(T(3,3n+1),\mathbb{F}) \longrightarrow H^{-4n-1,j}(D,\mathbb{F}) \longrightarrow 0$$

and we conclude $H^{-4n-2,j}(D,\mathbb{F}) \cong H^{-4n-2,j+1}(T(3,3n+1),\mathbb{F}) = 0$ and $H^{-4n-1,j}(D,\mathbb{F}) \cong H^{-4n-1,j+1}(T(3,3n+1),\mathbb{F}) = 0$, where calculations of $H(T(3,3n+1),\mathbb{F})$ are given in Turner's [14].

It remains to check the portion of the long exact sequence containing $i = -4n-2, -4n-1$ in the cases $j = -12n-6, -12n-4$, which are displayed below.

$$\begin{aligned} 0 \longrightarrow H^{-4n-2,-12n-3}(T(3,3n+1),\mathbb{F}) &\longrightarrow H^{-4n-2,-12n-4}(D,\mathbb{F}) \longrightarrow H^{0,1}(U,\mathbb{F}) \\ \xrightarrow{\delta_*} H^{-4n-1,-12n-3}(T(3,3n+1),\mathbb{F}) &\longrightarrow H^{-4n-1,-12n-4}(D,\mathbb{F}) \longrightarrow 0. \end{aligned}$$

$$\begin{aligned} 0 \longrightarrow H^{-4n-2,-12n-5}(T(3,3n+1),\mathbb{F}) &\longrightarrow H^{-4n-2,-12n-6}(D,\mathbb{F}) \longrightarrow H^{0,-1}(U,\mathbb{F}) \\ \xrightarrow{\delta_*} H^{-4n-1,-12n-5}(T(3,3n+1),\mathbb{F}) &\longrightarrow H^{-4n-1,-12n-6}(D,\mathbb{F}) \longrightarrow 0. \end{aligned}$$

In [14], Turner also calculates

$$\begin{aligned} H^{-4n-2,-12n-3}(T(3,3n+1),\mathbb{F}) &= 0 \\ H^{-4n-2,-12n-5}(T(3,3n+1),\mathbb{F}) &= 0 \\ H^{-4n-1,-12n-3}(T(3,3n+1),\mathbb{F}) &= \mathbb{F} \\ H^{-4n-1,-12n-5}(T(3,3n+1),\mathbb{F}) &= \mathbb{F}. \end{aligned}$$

And of course $H^{0,\pm 1}(U,\mathbb{F}) = \mathbb{F}$ for any field $\mathbb{F}$. Thus we have exact sequences

$$0 \longrightarrow H^{-4n-2,-12n-4}(D,\mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1,-12n-4}(D,\mathbb{F}) \longrightarrow 0 \tag{12}$$

and

$$0 \longrightarrow H^{-4n-2,-12n-6}(D,\mathbb{F}) \longrightarrow \mathbb{F} \xrightarrow{\delta_*} \mathbb{F} \longrightarrow H^{-4n-1,-12n-6}(D,\mathbb{F}) \longrightarrow 0 \tag{13}$$

From (12) and (13) it follows that each of the groups $H^{-4n-1,-12n-6}(D,\mathbb{F})$, $H^{-4n-1,-12n-4}(D,\mathbb{F})$, $H^{-4n-2,-12n-6}(D,\mathbb{F})$ and $H^{-4n-2,-12n-4}(D,\mathbb{F})$ is isomorphic to either $\mathbb{F}$ or the trivial group. We argue that all four of them are isomorphic to $\mathbb{F}$. By a straightforward calculation using Proposition 4.3 of Lee's [6], we find that $\dim_{\mathbb{F}}(Lee^{-4n-2}(\Delta^{2n+1},\mathbb{F})) = 2$. Using Turner's calculation of the homology of $T(3,3n+1)$, the long exact sequence (11) gives $H^{-4n-2,j}(\Delta^{2n+1},\mathbb{F}) \cong H^{-4n-2,j+1}(T(3,3n+1),\mathbb{F}) = 0$ for $j \neq -12n-6, -12n-4$. Therefore, $\dim_{\mathbb{F}} H^{-4n-2,*}(\Delta^{2n+1},\mathbb{F}) \leq 2$. Since the Lee spectral sequence has E^1 page the Khovanov homology and converges to Lee homology, we must also have $\dim_{\mathbb{F}} H^{-4n-2,*}(\Delta^{2n+1},\mathbb{F}) \geq 2$, and so it follows that $H^{-4n-2,-12n-6}(D,\mathbb{F}) = \mathbb{F}$ and $H^{-4n-2,-12n-4}(D,\mathbb{F}) = \mathbb{F}$. Finally, the non-triviality of these two groups together with (12) and (13) imply that $H^{-4n-1,-12n-4}(D,\mathbb{F}) = \mathbb{F}$ and $H^{-4n-2,-12n-6}(D,\mathbb{F}) = \mathbb{F}$.

Theorem 3.2. *For $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ for any prime p, and any $n \geq 0$,*

$$H(\Delta^{2n+1},\mathbb{F}) \cong H(T(3,3n+1),\mathbb{F})\{-1\} \oplus H(U,\mathbb{F})[-4n-2]\{-12n-5\}.$$

Corollary 3.3. *Let L be a link in Ω_3. The Khovanov homology $H(L)$ of L contains no $\mathbb{Z}_{p^r}$-torsion for $p \neq 2$.*

4 Spectral Sequences and Even Torsion

In the previous sections we showed that the Khovanov homology of closed 3-braids may contain only $\mathbb{Z}_{2^r}$-torsion. In this section we will show that all torsion in the Khovanov homology of links of type $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 is in fact $\mathbb{Z}_2$-torsion. To achieve this result we will analyze the interaction between two spectral sequences. The first of these sequences, the Bockstein spectral sequence, arises from an exact couple. The second, which we will refer to as the Turner spectral sequence, arises as the spectral sequences associated to a double complex. We begin by outlining the construction of the Bockstein and Turner spectral sequences, along with a variant of Khovanov homology called Bar-Natan homology [1] which may be viewed as a version of Lee homology [6] over $\mathbb{Z}_2$ coefficients.

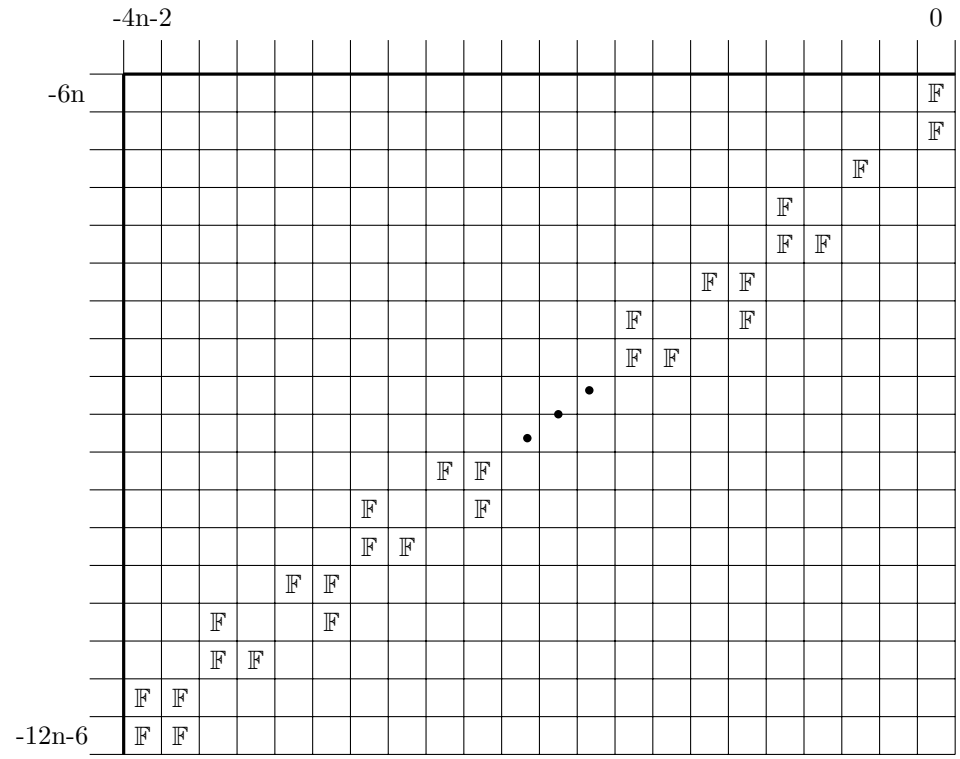
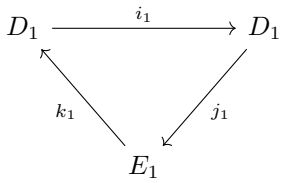


Figure 7: Khovanov homology of the braid closure of Δ^{2n+1} over $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ for p an odd prime.

4.1 The Bockstein Spectral Sequence

The Bockstein spectral sequence arises as the spectral sequence associated to an exact couple.

Definition 4.1. *Let D_1 and E_1 be R-modules, and let $i_1 : D_1 \to D_1$, $j_1 : D_1 \to E_1$ and $k_1 : E_1 \to D_1$ be R-module homomorphisms satisfying* $\ker(i_1) = \mathrm{Im}(k_1)$, $\ker(j_1) = \mathrm{Im}(i_1)$ *and* $\ker(k_1) = \mathrm{Im}(j_1)$. *The pentuple $(D_1, E_1, i_1, j_1, k_1)$ is called an exact couple. Succinctly, an exact couple is an exact diagram of R-modules and homomorphisms of the form*



The map $d_1 : E_1 \to E_1$ defined by $d_1 := j_1 \circ k_1$ satisfies $d_1 \circ d_1 = (j_1 \circ k_1) \circ (j_1 \circ k_1) = j_1 \circ (k_1 \circ j_1) \circ k_1 = 0$ and so defines a differential on E_1. Define $E_2 = H(E_1, d_1)$ to be the homology of the pair (E_1, d_1), and define $D_2 = \mathrm{Im}(i_1 : D_1 \to D_1)$. Also, define maps $i_2 : D_2 \to D_2$, $j_2 : D_2 \to E_2$ and $k_2 : E_2 \to D_2$ by $i_2 = i_1|_{D_2}$, $j_2(i_1(a)) = j_1(a) + d_1 E_1$ and $k_2(e + d_1 E_1) = k_1(e)$. The resulting pentuple $(D_2, E_2, i_2, j_2, k_2)$ is another an exact couple [8, Proposition 2.7]. Iterating this process produces a sequence $(E_r, d_r)_{r \geq 1}$ called the *spectral sequence associated to the exact couple* $(D_1, E_1, i_1, j_1, k_1)$.

Given a chain complex (C, d), let us denote its integral homology by $H(C)$ and its homology over coefficients in $\mathbb{Z}_2$ by $H(C; \mathbb{Z}_2)$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\operatorname{red} p} \mathbb{Z}_p \longrightarrow 0 \tag{14}$$

where $\times p$ is multiplication by p, and $\operatorname{red} p$ is reduction modulo p. Tensoring a chain complex (C, d) with (14) yields a short exact sequence of chain complexes

$$0 \longrightarrow C \xrightarrow{\times p} C \xrightarrow{\operatorname{red} p} C \otimes \mathbb{Z}_p \longrightarrow 0$$

The associated long exact sequence in homology can be viewed as an exact couple

$$\begin{array}{ccc} H(C) & \xrightarrow{\quad\times p\quad} & H(C) \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle \operatorname{red} p} & \\ & H(C;\mathbb{Z}_p) & \end{array} \tag{15}$$

where ∂ is the connecting homomorphism.

Definition 4.2. *Let p be a positive prime integer. The spectral sequence $(E_B^r, d_B^r)_{r\geq 1}$ associated to the exact couple (15) is called the $\mathbb{Z}_p$-Bockstein spectral sequence.*

The following properties of the Bockstein spectral sequence will be of great importance for our purposes, the proofs of which can be found in [8].

(B1) The first page is $(E_B^1, d_B^1) = (H(C;\mathbb{Z}_p), \beta)$ where $\beta = (\operatorname{red} p) \circ \partial$.

(B2) The limiting page is $E_B^\infty = (H(C)/\text{Torsion}) \otimes \mathbb{Z}_p$.

(B3) If the Bockstein spectral sequence collapses on the E^k page, that is, if $E_B^k = E_B^\infty$, then $H(C)$ contains no $\mathbb{Z}_{p^r}$-torsion for $r \geq k$.

(B4) If $C = C^{*,*}$ is a bigraded complex with degree preserving differential, then the differentials d_B^r have bidegree $(1, 0)$.

In order to achieve our stated goal of showing that links of type $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 may contain only $\mathbb{Z}_2$-torsion, it suffices by property (B3) to show that the $\mathbb{Z}_2$-Bockstein spectral sequence associated to the Khovanov complex of such links collapses on the E^2 page. To this end, we will consider its interaction with a second spectral sequence, the Turner spectral sequence.

4.2 Bar-Natan Homology and The Turner Spectral Sequence

In [1], Bar-Natan gives a variant of $\mathbb{Z}_2$-Khovanov homology, now called Bar-Natan homology. The construction of Bar-Natan homology follows that of section 3.1 but with a different choice of differential, d_b, defined with multiplication and comultiplication given by

$$m_B(1 \otimes 1) = 1 \quad m_B(1 \otimes X) = X \quad m_B(X \otimes 1) = X \quad m_B(X \otimes X) = X$$
$$\Delta_B(1) = 1 \otimes X + X \otimes 1 + 1 \otimes 1 \quad \Delta_B(X) = X \otimes X$$

and with unit and counit structures given by

$$i(1) = 1 \quad \epsilon(1) = 0 \quad \epsilon(X) = 1.$$

The resulting singly graded homology theory is a link invariant denoted $BN^*(L)'$ and Turner shows the following fact in [15, Theorem 3.1].

Lemma 4.3. *Let L be an oriented link with k components. Then, $\dim_{\mathbb{Z}_2}(BN^*(L)') = 2^k$. Specifically, if the components of L are $L_1, L_2, \ldots, L_k$, then*

$$\dim_{\mathbb{Z}_2}(BN^i(L)') = \#\left\{ E \subset \{1,2,\ldots,k\} \;\middle|\; 2 \times \sum_{l\in E, m\in E^c} lk(L_l, L_m) = i \right\}$$

where $lk(L_l, L_m)$ is the linking number between L_l and L_m.

Example 4.4. *Let n be an integer.*

(1) $T(3,3n)$ is a three component link with $BN^0(T(3,3n))' = \mathbb{Z}_2^2$, $BN^{-4n}(T(3,3n))' = \mathbb{Z}_2^6$ and $BN^i(T(3,3n))' = 0$ for $i \neq 0, -4n$.

(2) $T(3,3n+1)$ is a knot with $BN^0(T(3,3n+1))' = \mathbb{Z}_2^2$ and $BN^i(T(3,3n+1))' = 0$ for $i \neq 0$.

(3) $T(3,3n-1)$ is a knot with $BN^0(T(3,3n-1))' = \mathbb{Z}_2^2$ and $BN^i(T(3,3n-1))' = 0$ for $i \neq 0$.

In [15] Turner defines a map d_T on the $\mathbb{Z}_2$-Khovanov complex in terms of Kauffman states and edge maps in the same way as the Khovanov differential d except with multiplication and comultiplication given by

$$m_T(1\otimes 1) = 0 \quad m_T(1\otimes X) = 0 \quad m_T(X\otimes 1) = 0 \quad m_T(X\otimes X) = X$$

$$\Delta_T(1) = 1\otimes 1 \quad \Delta_T(X) = 0.$$

and the map d_T satisfies $d_T^2 = 0$ and $d\circ d_T + d_T \circ d = 0$.

Definition 4.5 (Turner, P)**.** *Let D be a diagram of a link L and $(C(D;\mathbb{Z}_2), d)$ be the $\mathbb{Z}_2$-Khovanov complex. The spectral sequence $(E_T^r, d_T^r)_{r\geq 1}$ associated to the double complex $(C(D;\mathbb{Z}_2), d, d_T)$ is called the Turner spectral sequence.*

We have the following properties of the Turner spectral sequence

(T1) The first page is $(E_T^1, d_T^1) = (H(L,\mathbb{Z}_2), d_T^*)$ where $d_T^* : H(L,\mathbb{Z}_2) \to H(L,\mathbb{Z}_2)$ is the induced map on homology.

(T2) Each map d_T^r is a differential of bidegree $(1, 2r)$.

(T3) The Turner spectral sequence converges to Bar-Natan homology: $E_T^r \implies BN^*(L)'$. In light of (T2), this simply means that taking the direct sum along the i-th column of the limiting page yields $BN^i(L)'$.

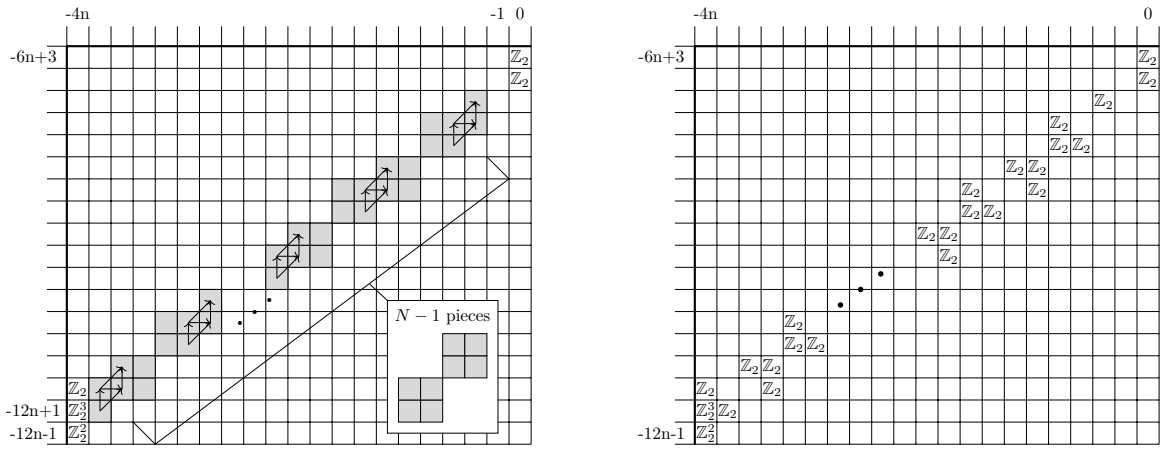
The next ingredient in the proof of Theorem 4.7 is a 'vertical' differential on the Khovanov complex due to Khovanov [5]. Let D be a diagram of a link L. A differential $\nu : C(D;\mathbb{Z}_2) \to C(D;\mathbb{Z}_2)$ is defined as follows. Recall that for a Kauffman state $D(I)$, the algebra $C(D(I))$ has $2^{|D(I)|}$ generators of the form $a_1\otimes a_2 \otimes \cdots \otimes a_{|D(I)|}$ where $a_i \in \{1, X\}$ and $|D(I)|$ is the number of circles in $D(I)$. For each Kauffman state we define a map $\nu_{D(I)} : C(D(I)) \to C(D(I))$ by sending a generator to the sum of all possible generators obtained by replacing a single X with a 1. For example, $X\otimes X\otimes X \mapsto 1\otimes X\otimes X + X\otimes 1\otimes X + X\otimes X\otimes 1$. We then extend $\nu_{D(I)}$ linearly to all of $C(D(I))$, and then to a map $\nu : C(D;\mathbb{Z}_2) \to C(D;\mathbb{Z}_2)$. The properties of this map relevant to our purposes are given below, the proofs of which can be found in [13].

(V1) ν is a differential of bidegree $(0, 2)$.

(V2) ν commutes with the Khovanov differential d, and so induces a map (differential) $\nu^* : H(L,\mathbb{Z}_2) \to H(L,\mathbb{Z}_2)$ on homology.

(V3) The complex $(H(L,\mathbb{Z}_2), \nu^*)$ is acyclic, that is, it has trivial homology.

Lemma 4.6 ([13], Lemma 3.2.A)**.** *Let L be a link. The Bockstein, Turner and vertical differentials on Khovanov homology $H(L,\mathbb{Z}_2)$ are related by $d_T^* = \beta\circ\nu^* + \nu^*\circ\beta$.*



(a) (b)

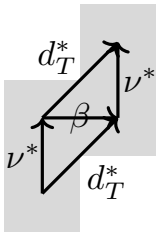
Figure 8: In (a) we have The $\mathbb{Z}_2$-Khovanov homology of the torus link $T(3,3n)$ and E^1 page of both the Bockstein and Turner spectral sequences. Each gray box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have the E^∞ page of the Bockstein spectral sequence on $T(3,3n)$.

4.3 Even Torsion in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$

We now have the tools to prove the main result of this paper.

Theorem 4.7. *The Khovanov homology for braids in $\Omega_0, \Omega_1, \Omega_2, \Omega_3$, contains only $\mathbb{Z}_2$-torsion.*

Proof. By Lemma 3.1, the 3-strand torus links have no $\mathbb{Z}_{p^r}$-torsion for $p \neq 2$ in Khovanov homology. It remains to show that the $\mathbb{Z}_2$-Bockstein spectral sequence collapses on the second page for every 3-strand torus link. We break the proof up into the cases $T(3,3n)$, $T(3,3n+1)$, $T(3,3n-1)$, and Δ^{2n+1}. In the $\mathbb{Z}_2$-Khovanov homology of each case, we identify certain occurances of the maps d_T^*, β, ν^* arranged as shown below.



(i) **Case** $T(3,3n)$: By Theorem 4.12 of [2] and the relationship between Khovanov homology and its reduced counterpart [5], $H(T(3,3n),\mathbb{Z}_2)$ is as given in Figure 8a. By properties (B1) and (T1), this constitutes the E^1 page of both the Bockstein and Turner spectral sequences for the link $T(3,3n)$. By property (B2) and Theorem 3.1(i) of [14], the E^∞ page of the Bockstein spectral sequence is as given in Figure 8b. Therefore, in order to show that the Bockstein spectral sequence collapses on the E^2 page, it suffices to show that each Bockstein map labeled β in Figure 8a is an isomorphism. To this end, recall from Example 4.4 that $BN^{-4n}(T(3,3n))' = \mathbb{Z}_2^6$ and $BN^0(T(3,3n))' = \mathbb{Z}_2^2$, while $BN^i(T(3,3n))' = 0$ otherwise. It follows by property (T3) that the E^∞ page of the Turner spectral sequence comprises those six $\mathbb{Z}_2$ summands in homological degree $i = -4n$ and the two $\mathbb{Z}_2$ summands in degree $i = 0$, which are unshaded in Figure 8a. Consequently, all groups in the range $-4n < i < 0$ must eventually be killed in the course of turning pages of the Turner spectral sequence. Keeping in mind that the

r-th Turner differential has bidegree $(1, 2r)$, let us examine, moving from left to right and starting with homological degree $i = -4n + 1$, how this killing must occur. First note that the two $\mathbb{Z}_2$ summands in degree $i = -4n + 1$ must cancel with the two summands in degree $i = -4n + 2$. Moreover, the only way for these four summands to be killed is for the maps $d_T^* : H^{-4n+1,-12+1}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+2,-12+3}(T(3, 3n), \mathbb{Z}_2)$ and $d_T^* : H^{-4n+1,-12+3}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+2,-12+5}(T(3, 3n), \mathbb{Z}_2)$ to be isomorphisms. Since $d_T^* = \beta \circ \nu^* + \nu^* \circ \beta$, and since the maps ν^* are isomorphisms by property (V3), it follows the Bockstein map $\beta : H^{-4n+1,-12n+3}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+2,-12n+3}(T(3, 3n), \mathbb{Z}_2)$ is an isomorphism. Next observe that, again due to degree considerations, the two $\mathbb{Z}_2$ summands in degree $i = -4n + 3$ must cancel with the two in degree $i = -4n + 4$. Then, by an argument similar to the above, the maps $d_T^* : H^{-4n+5,-12n+7}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+6,-12n+9}(T(3, 3n), \mathbb{Z}_2)$ and $d_T^* : H^{-4n+5,-12n+9}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+6,-12n+11}(T(3, 3n), \mathbb{Z}_2)$ are isomorphisms, and consequently so is the Bockstein map $\beta : H^{-4n+5,-12n+9}(T(3, 3n), \mathbb{Z}_2) \to H^{-4n+6,-12n+9}(T(3, 3n), \mathbb{Z}_2)$. Repeating this process, we find that each Bockstein map β in Figure 8a is an isomorphism. Hence the Bockstein spectral sequence collapses on the E^2 page and $H(T(3, 3n))$ contains no $\mathbb{Z}_{2^k}$-torsion for $k \geq 2$.

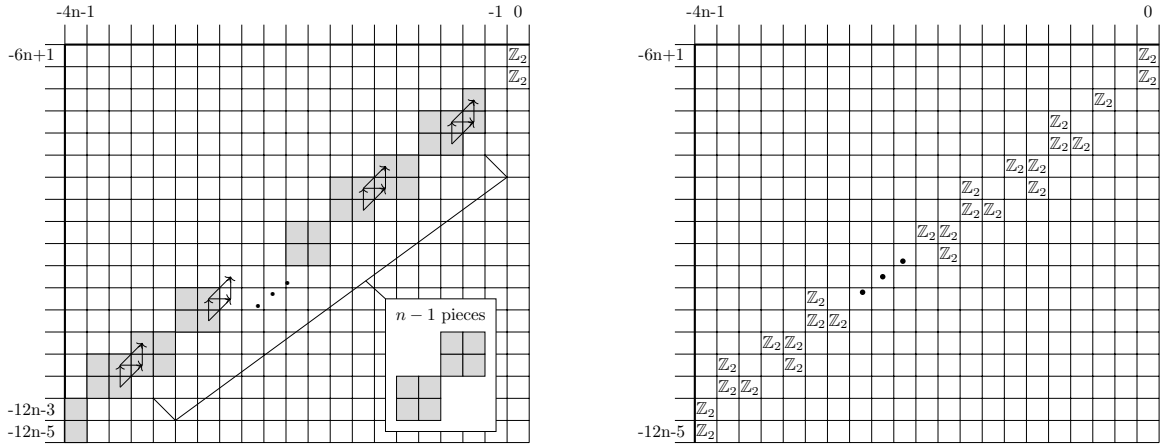


Figure 9: In (a) we have $H(T(3, 3n + 1), \mathbb{Z}_2)$: The E^1 page of both the Bockstein and Turner spectral sequences. Each gray box represents a copy of $\mathbb{Z}_2$ which is killed in the Turner spectral sequence. In (b) we have E^∞ page of the $\mathbb{Z}_2$-Bockstein spectral sequence for $T(3, 3n + 1)$.

(ii) **Case** $T(3, 3n + 1)$: By Theorem 4.12 of [2] and the relationship between Khovanov homology and its reduced counterpart [5], $H(T(3, 3n + 1), \mathbb{Z}_2)$ is as given in Figure 9a, and constitutes the the E^1 page of both the Turner and $\mathbb{Z}_2$-Bockstein spectral sequences. By property (B2) and Theorem 3.1(ii) of [14], the E^∞ page of the Bockstein spectral sequence is as given in Figure 9b. Therefore, in order to show that the Bockstein spectral sequence collapses on the E^2 page, it suffices to show that each map labeled β in Figure 9a is an isomorphism: As noted in Example 4.4, $BN^0(T(3, 3n + 1))' = \mathbb{Z}_2^2$ and $BN^i(T(3, 3n + 1))' = 0$ for $i \neq 0$. It follows by property (T3) that the E^∞ page of the Turner spectral sequence comprises the two $\mathbb{Z}_2$ summands in Figure 9a in homological grading $i = 0$. Consequently, all groups in the range $-4n < i < 0$ are killed in the course of turning pages of the Turner spectral sequence. First note that the two $\mathbb{Z}_2$ summands in degree $i = -4n - 1$ must cancel with the two in degree $i = -4n$. Next observe that the two $\mathbb{Z}_2$ summands in degree $i = -4n + 1$ must cancel with the two in degree $i = -4n + 2$, and moreover that the only way for these four summands to be killed is for the maps $d_T^* : H^{-4n+1,-12n-1}(T(3, 3n + 1), \mathbb{Z}_2) \to H^{-4n+2,-12n+1}(T(3, 3n + 1), \mathbb{Z}_2)$ and $d_T^* : H^{-4n+1,-12n+1}(T(3, 3n + 1), \mathbb{Z}_2) \to H^{-4n+2,-12n+3}(T(3, 3n + 1), \mathbb{Z}_2)$ to be isomorphisms. Since $d_T^* = \beta \circ \nu^* + \nu^* \circ \beta$, and since the maps ν^* are isomorphisms by property (V3), it follows that the Bockstein map $\beta : H^{-4n+1,-12n+1}(T(3, 3n + 1), \mathbb{Z}_2) \to H^{-4n+2,-12n+1}(T(3, 3n + 1), \mathbb{Z}_2)$ is

also an isomorphism. Repeating these two steps, we find that each of the Bockstein maps labeled β in Figure 9a is an isomorphism. Hence the Bockstein spectral sequence collapses on the E^2 page and $H(T(3,3n+1),\mathbb{Z}_2)$ contains no $\mathbb{Z}_{2^k}$-torsion for $k \geq 2$.

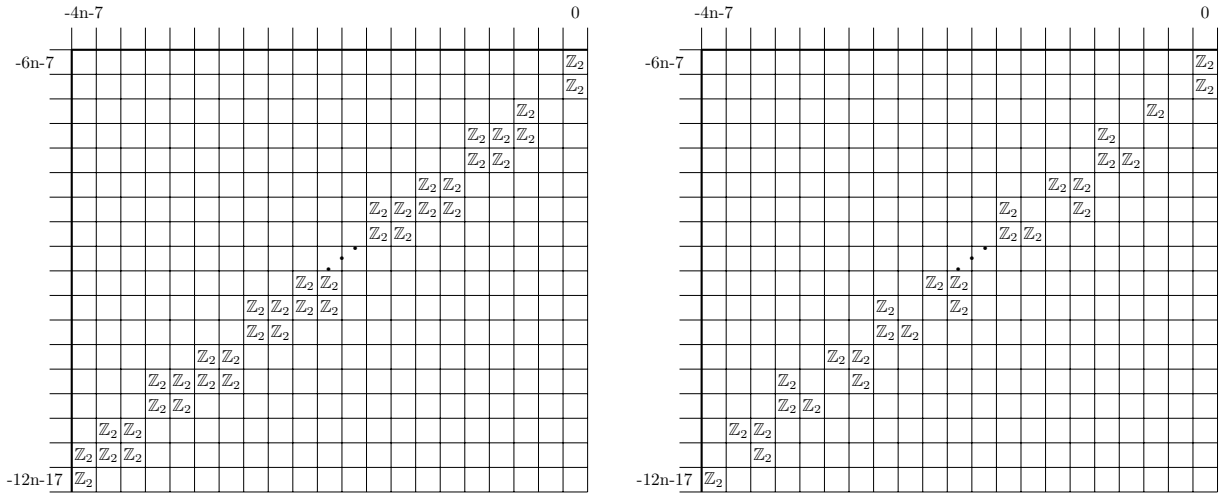


Figure 10: In (a) we have Khovanov homology of $T(3,3n+2)$ over $\mathbb{Z}_2$, and E^1 page of both the Bockstein and Turner spectral sequences. In (b) we have the E^∞ page of the $\mathbb{Z}_2$-Bockstein spectral sequence for $T(3,3n+2)$.

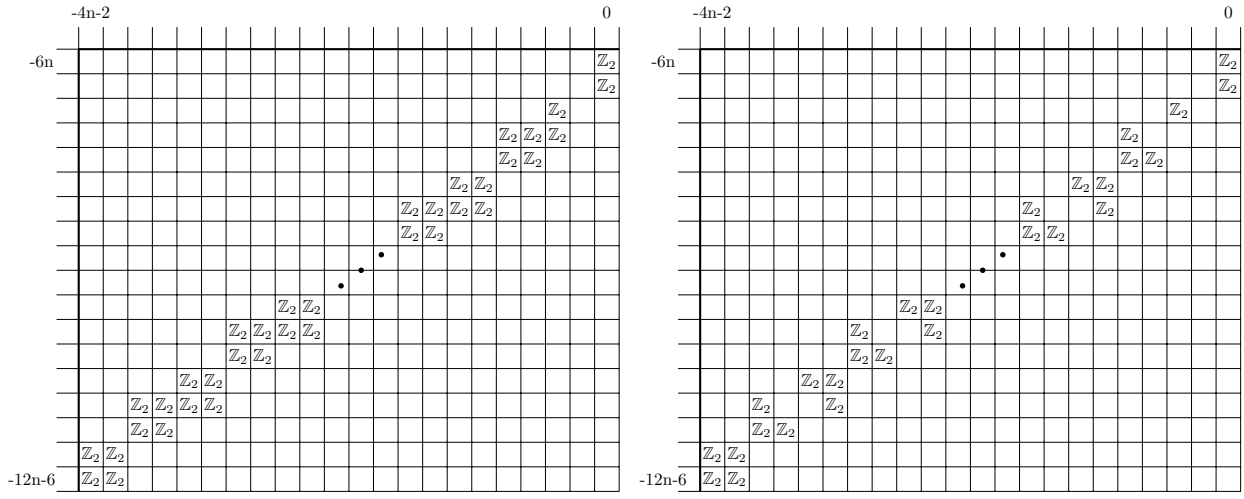


Figure 11: In (a) we have the $\mathbb{Z}_2$-Khovanov homology of the braid closure of Δ^{2n+1}, and E^1 page of both the Bockstein and Turner spectral sequences. In (b) we have the E^∞ page of the $\mathbb{Z}_2$-Bockstein spectral sequence for the braid closure of Δ^{2n+1}.

(iii) **Case** $T(3,3n+2)$: This case follows by the same line of argument as in (ii) (see figures 10a and 10b).

(iv) **Case Δ^{2n+1}**: From (12) and (13) it follows that each of the groups

$$H^{-4n-1,-12n-5\pm1}(D,\mathbb{Z}_2),\ H^{-4n-2,-12n-5\pm1}(D,\mathbb{Z}_2)$$

is isomorphic to either $\mathbb{Z}_2$ or the trivial group. We argue that all four of them are isomorphic to $\mathbb{Z}_2$. Using Theorem 3.1 of Turners's [15], we find that $\dim_{\mathbb{Z}_2}(BN^{-4n-2}(\Delta^{2n+1},\mathbb{F})') = 2$. Using Benheddi's calculation [2] of the $\mathbb{Z}_2$-Khovanov homology of $T(3,3n+1)$, shown in Figure 11a, the long exact sequence (11) gives $H^{-4n-2,j}(\Delta^{2n+1},\mathbb{Z}_2) \cong H^{-4n-2,j+1}(T(3,3n+1),\mathbb{Z}_2) = 0$ for $j \neq -12n-6, -12n-4$. Therefore, $\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1},\mathbb{Z}_2) \leq 2$. Since the Turner spectral sequence has E^1 page the $\mathbb{Z}_2$-Khovanov homology and converges to Bar-Natan homology, we must also have $\dim_{\mathbb{Z}_2} H^{-4n-2,*}(\Delta^{2n+1},\mathbb{Z}_2) \geq 2$, and so it follows that $H^{-4n-2,-12n-6}(D,\mathbb{Z}_2) = \mathbb{Z}_2$ and $H^{-4n-2,-12n-4}(D,\mathbb{Z}_2) = \mathbb{Z}_2$. Finally, the non-triviality of these two groups together with (12) and (13) imply that $H^{-4n-1,-12n-4}(D,\mathbb{Z}_2) = \mathbb{Z}_2$ and $H^{-4n-2,-12n-6}(D,\mathbb{Z}_2) = \mathbb{Z}_2$.

Figure 11a shows the first page of the Turner and Bockstein spectral sequences for Δ^{2n+1} over $\mathbb{Z}_2$. Figure 5b implies the infinity page of the Bockstein spectral sequence is given in Figure 11b. An argument similar to the one presented in the cases first three applies here. The only difference is that Δ^{2n+1} is a 2-component link and thus the Turner spectral sequence has all terms in homological degree $i = -4n-7$ and 0 survive to the infinity page.

□

Corollary 4.8. *For any $n \geq 0$,*

$$H(\Delta^{2n+1}) \cong H(T(3,3n+1))\{-1\} \oplus H(U)[-4n-2]\{-12n-5\}.$$

Corollary 4.9. *The integral Khovanov homology of links in classes $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ are given in Figures 4a, 4b, 5a, and 5b.*

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