A Broken Circuit Model for Chromatic Homology

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Abstract

In 2004, Helme-Guizon and Rong presented a categorification of the chromatic polynomial for graphs using a construction analogous to Khovanov's categorification of the Jones polynomial. In this categorification, the chromatic polynomial is upgraded to a homological object and the deletion contraction rule lifts to a long exact sequence. We will explore how Whitney's broken circuit theorem manifests itself at the categorified level. This was also done independently by Sazdanovič and Yip using a different method.

1. The Chromatic Polynomial

Given a graph $G = (V, E)$, a *proper n-coloring* is a map $c : V \to \{1, 2, \dots, n\}$ for which no adjacent vertices get the same 'color'. Let $P_G(x)$ denote the number of proper x-colorings of G. It turns out that

$$P_G(x) = \sum_{S \in 2^E} (-1)^{|S|} x^{k(S)}$$

where $k(S)$ is the number of connected components in the spanning subgraph with edge set S. In particular, $P_G(x)$ is a polynomial. Let the edge set E have a fixed ordering $E = \{e_1, \dots e_m\}$. A *broken circuit* in G is $C - e$ where C is a cycle in G and e is the largest labeled edge in C. Let NBC be the set of all $S \in 2^E$ which contain no broken circuits and let $\mathrm{BC} = 2^E - \mathrm{NBC}$ be its complement.

2. The Broken Circuit Theorem

Whitney used an inductive argument to show that all contributions from BC cancel in $P_G(x)$. That is,

$$P_G(x) = \sum_{S \in \mathrm{NBC}} (-1)^{|S|} x^{k(S)}$$

A few proofs of this fact exist. Here we will borrow an involution $i : \mathrm{BC} \to \mathrm{BC}$ from the proof of Blass and Sagan (see 'proof of lemma' in the right column).

3. The Categorification Method

The *graded dimension* of a graded vector space $V = \oplus_{i \in \mathbb{Z}} V_i$ is the power series $q \dim V = \sum_{i \in \mathbb{Z}} q^i \dim V_i$. Given a cochain complex $C = \oplus_{i \in \mathbb{Z}} C^i$ of graded vector spaces, the *graded Euler characteristic* is

$$\chi_q(C) = \sum_{i \in \mathbb{Z}} (-1)^i q \dim C^i.$$

If the differential in C preserves grading, then cohomology preserves χ_q. That is, $\chi_q(C) = \chi_q(H^*(C))$. The idea is to lift the sum formula for $P_G(x)$ to a chain complex of graded vector spaces whose graded Euler characteristic is the chromatic polynomial.

4. The Chromatic Cohomology

For purposes of categorification it is convenient to replace the variable x with $1 + q$. Let A denote the graded $\mathbb{Z}_2$-algebra $A = \mathbb{Z}_2[x]/(x^2)$ so $q \dim A = 1 + q$. For $S \in 2^E$ define $A_S = A^{\otimes k(S)}$ and set

$$C(G) = \oplus_{S \in 2^E} A_S \qquad C^i(G) = \oplus_{|S|=i} A_S$$

The differential on $C(G)$ arises from edges in the Hasse diagram of 2^E and the multiplication in A. The idea is to multiply tensor factors corresponding to connected components when the transition from S to $S + e$ joins two components (see Figure 1).

5. The Picture You Should Have in Mind

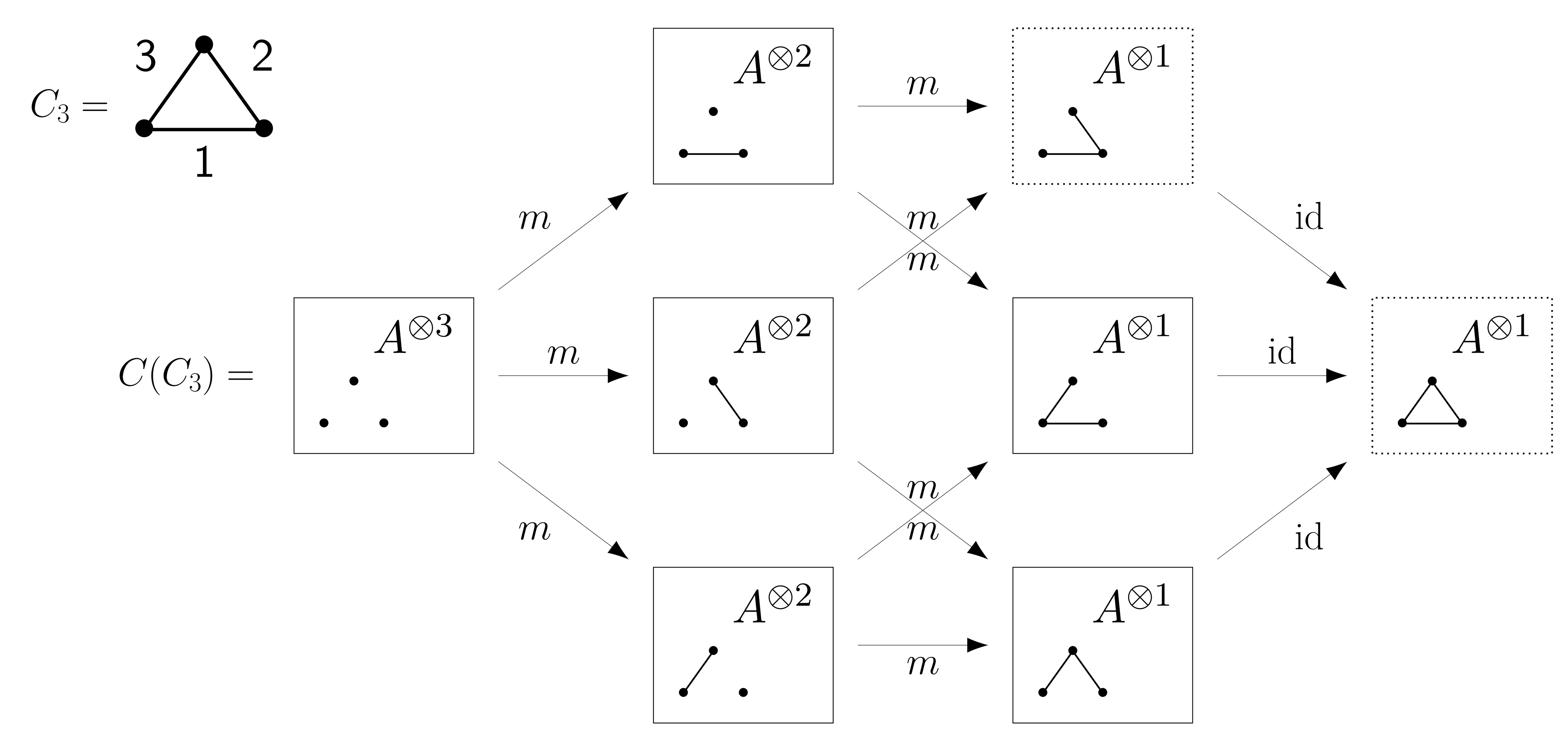

Figure 1: The chromatic chain complex for the 3-cycle C_3. Subgraphs containing broken circuits have a dotted outline.

6. The Fundamental Lemma

The set $\mathrm{BC} \subset 2^E$ has a complete acyclic matching (see Figure 2) for which each matched pair has the same number of connected components.

7. The Main Theorem

The subcomplex $C_{\mathrm{BC}}(G)$ of $C(G)$ generated by $\mathrm{BC} \subset 2^E$ is acyclic and therefore $C(G)$ has the same cohomology as $C_{\mathrm{NBC}}(G) = \oplus_{S \in \mathrm{NBC}} A_S$.

8. The Main Idea

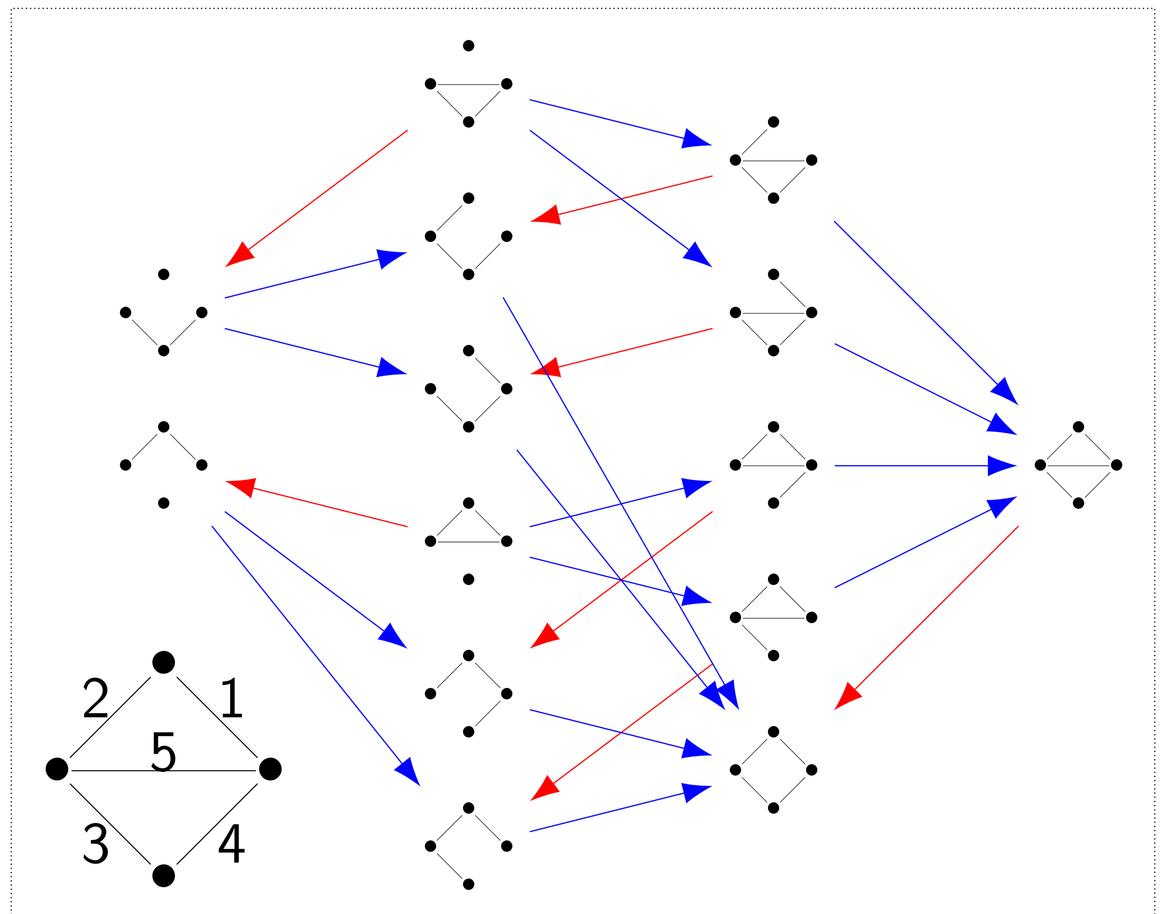

Figure 2: The broken circuit complex for the graph pictured above with an acyclic matching as per the Lemma. Here, 'acyclic' means that upon reversing all edges in the matching, there are no directed cycles.

Proof of Lemma. We define an involution $i : \mathrm{BC} \to \mathrm{BC}$. Given $S \in \mathrm{BC}$, let e be the maximum over all $e \in E$ such that S contains a broken circuit $C - e$. Define $i(S)$ to be $S + e$ if $e \notin S$ or $S - e$ if $e \in S$. It follows that $i(i(S)) = S$ and thus the orbits of i form a complete matching M on BC. To show this is acyclic, it suffices to show there is a linear extension of BC in which matched edges become cover relations. For $e = (S \lessdot T) \in M$ let $d(e) = S$. Fix a total ordering on $d(M) = \{d(e) \mid e \in M\}$ by setting $U \le V$ if $|U| < |V|$ or if $|U| = |V|$ and U is lexicographically larger than V (that is, order ranks by reverse lexicographic order). Fix the notation $M = \{(S_1 \lessdot T_1), \dots (S_n \lessdot T_n)\}$ with the S_i ordered as indicated above. One can check that $S_1, T_1, S_2, T_2, \dots, S_n, T_n$ is the desired linear extension. Thus M is indeed acyclic.

Proof of main theorem. This follow immediately from the lemma and the main theorem of algebraic Morse theory since all matched edges are isomorphisms in $C(G)$.